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Regulation-triggered adaptive control of a hyperbolic PDE-ODE model with boundary interconnections

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Summary
We present a certainty equivalence-based adaptive boundary control scheme with a regulation-triggered batch least-squares identifier, for a heterodirectional transport partial differential equation-ordinary differential equation (PDE-ODE) system where the transport speeds of both transport PDEs are unknown. We use a nominal controller which is fed piecewise-constant parameter estimates from an event-triggered parameter update law that applies a least-squares estimator to data "batches" collected over time intervals between the triggers. A parameter update is triggered by an observed growth in the norm of the PDE state. The proposed triggering-based adaptive control guarantees: (1) the absence of a Zeno phenomenon; (2) parameter estimates are convergent to the true values in finite time (from most initial conditions); (3) exponential regulation of the plant states to zero. The effectiveness of the proposed design is verified by a numerical example.

KEYWORDS
adaptive control, backstepping, event-triggered control, hyperbolic PDEs, least-squares identifier.

1 | INTRODUCTION

1.1 | Control of coupled transport PDEs

Systems of transport partial differential equations (PDEs) appear in many physical models, including road traffic,[1-4] water level dynamics,[5-7] and flow of fluids in transmission lines.[8,9] In the past 10 years, many authors have contributed to boundary control of the coupled transport PDEs. The basic boundary stabilization problem of 2×2 coupled linear transport PDEs by backstepping was addressed in References 10,11, which was further extended to boundary control of an $n+1$ system in Reference 12. For a more general coupled transport PDE system, where the number of PDEs in either direction is arbitrary, a boundary stabilization law was first designed by backstepping in Reference 13. Boundary control of coupled transport PDEs connected with ordinary differential equations (ODEs) at the uncontrolled boundary was studied in References 14-17.

Through the Riemann transformations, wave PDEs can be converted to 2×2 hyperbolic PDEs.[18-21] Therefore, in addition to the applications to the traffic and water-level dynamics, the boundary control design for coupled first-order hyperbolic PDEs has also been applied to wave PDE-modeled dynamics, such as oil drilling,[19] cable elevators,[22] and deep-sea construction vessels.[23]

1.2 | Adaptive control of PDEs

Based on approximated reduced-order models, learning-based state estimation and stabilization for PDEs with parameter uncertainties are presented in References 24,25, respectively. For fully model-based adaptive control[26] of PDEs without model approximation, three traditional schemes are the Lyapunov design, the passivity-based design, and the swapping design,[27] which were initially developed for ODEs in Reference 28. Using the three design methods, adaptive control designs were proposed for parabolic PDEs in References 29-31. For adaptive control of hyperbolic PDEs, many results, based on the three conventional methods, have been introduced in References 19,32-43. An adaptive control application to congested traffic is shown in Reference 2. As in all conventional adaptive control, the adaptive control designs achieve only asymptotic convergence of the plant states, without being guaranteed to identify exactly the true parameters.

Recently, a new adaptive scheme, using a regulation-triggered batch least-squares identifier (BaLSI), was introduced in References 44,45, which has at least two significant advantages over the traditional adaptive approaches: guaranteeing exponential regulation of the states to zero, as well as finite-time convergence of the estimates to the true values. An application of BaLSI to a two-link manipulator, which is modeled by a highly nonlinear system and subject to four parametric uncertainties, is shown in Reference 46. Regarding PDEs, this method has been successfully applied in adaptive control of a parabolic PDE where the unknown parameters are the reaction coefficient and the high-frequency gain.[47] Using the least-squares identifier updated in a sequence of times with fixed intervals, the backstepping adaptive boundary control design of a first-order hyperbolic PDE where the transport speed is allowed to be unknown, was first proposed in Reference 48, and extended to the case with a spatially varying coefficient in Reference 49. Extending the result in Reference 50 with removing the restriction of nonzero initial conditions of PDEs, in this paper we develop a BaLSI-based backstepping adaptive boundary controller for a heterodirectional transport PDE-ODE system, where both of the two transport speeds are unknown.

1.3 | Contributions

- In this paper, we design an adaptive certainty-equivalence controller with regulation-triggered batch least-squares identification for a coupled hyperbolic PDE-ODE system where the unknown parameters are transport speeds. It would be appropriate to regard this paper as the hyperbolic equivalent of the paper[47] for a parabolic PDE, where the unknown parameters are the reaction coefficient and the high-frequency gain.
- Compared with previous adaptive control results for coupled transport PDEs,[32-35] where the transport speeds or transport delays are required to be known, and only the asymptotic convergence to zero of the plant states is achieved, our design deals with unknown transport speeds (only bounds being known) with finite-time exact estimations (from most initial conditions), and achieves exponential regulation of the plant states to zero. To permit the transport speeds to be unknown, we require the full PDE state to be measured (as might be feasible in congested traffic applications on fully instrumented freeways).
- As compared to References 48,49 which deal with adaptive control of a first-order hyperbolic PDE with an uncertain in-domain source parameter and an uncertain transport speed, where the parameter identifiers were employed with updates at a sequence of times separated by fixed intervals, we design here an adaptive controller for a heterodirectional first-order hyperbolic PDE-ODE system, where both of the two transport speeds are unknown, and the identifier is updated at a sequence of nonequidistant times, which are determined by an event trigger that is activated based on the progress of the regulation of the plant's PDE and ODE states.

1.4 | Organization

The problem formulation is shown in Section 2. The nominal control design is presented in Section 3. The design of the regulation-triggered adaptive controller is proposed in Section 4. The main results, including the parameter convergence, the exclusion of the zeno phenomenon, and exponential regulation of the states, are proved in Section 5. The effectiveness of the proposed design is illustrated with simulation in Section 6. The conclusion, are given in Section 7.

1.5 Notation

We adopt the following notation.

- The symbol Z_+ denotes the set of all nonnegative integers and $\mathbb{R}_+ := [0, +\infty)$.
- Let $U \subseteq \mathbb{R}^n$ be a set with non-empty interior and let $\Omega \subseteq \mathbb{R}$ be a set. By $C^0(U; \Omega)$, we denote the class of continuous mappings on U, which take values in Ω.
- We use the notation $\mathbb{N}$ for the set $\{1, 2, \dots\}$, that is, the natural numbers without 0.
- For an $I \subseteq \mathbb{R}_+$, the space $C^0(I; L^2(0, 1))$ is the space of continuous mappings $I \ni t \mapsto u[t] \in L^2(0, 1)$.
- Let $u : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}$ be given. We use the notation $u[t]$ to denote the profile of u at certain $t \geq 0$, that is, $(u[t])(x) = u(x, t)$, for all $x \in [0, 1]$.

2 PROBLEM FORMULATION

In this paper we consider the class of plants

$$\dot{\zeta}(t) = (a - q_1 bc)\zeta(t) + b(q_2 + q_1 p)w(0, t), \quad t \geq 0, \tag{1}$$

$$z_t(x, t) = -q_1 z_x(x, t), \quad x \in [0, 1],\ t \geq 0, \tag{2}$$

$$w_t(x, t) = q_2 w_x(x, t), \quad x \in [0, 1],\ t \geq 0, \tag{3}$$

$$z(0, t) = c\zeta(t) - pw(0, t), \quad t \geq 0, \tag{4}$$

$$w(1, t) = \frac{\bar{c}}{q_2} U(t) + \frac{q_1}{q_2} z(1, t), \quad t \geq 0 \tag{5}$$

with initial conditions $w(x, 0) = w_0(x)$ for $x \in [0, 1)$, $z(x, 0) = z_0(x)$ for $x \in (0, 1]$, $\zeta(0) = \zeta_0$, where $\zeta(t)$ is a scalar ODE state and scalar $z(x, t), w(x, t)$ are PDE states. The boundary condition (5) contains the control input $U(t)$. The class of (1)–(4) is motivated by a wave PDE converted to Riemann variables. It is through such a transformation process that possibly unmotivated–looking coefficients $a - q_1 bc$ and $b(q_2 + q_1 p)$ in (1) arise.

It is the parameters q_1 and q_2, which appear both as transport speeds and in the ODE (1) and the boundary condition (5) that we consider unknown. The speed q_2 is arbitrary and, of course, positive. The constants a, b, c, p are arbitrary and positive as well. The constant $\bar{c}$ is arbitrary and nonzero.

To make the problem as nontrivial as we can within this class, we only consider the case where the ODE (1) is unstable, with $a - q_1 bc > 0$, that is, the case where the unknown propagation speed q_1 satisfies

$$0 < q_1 < \frac{a}{bc}. \tag{6}$$

The unknown transport speeds q_1, q_2 are assumed to have known upper bounds $\overline{q}_1 > 0, \overline{q}_2 > 0$ and lower bounds $\underline{q}_1 > 0, \underline{q}_2 > 0$, respectively. The bounds $\underline{q}_2, \overline{q}_2$ are arbitrary, in addition to satisfying the obvious relation $\underline{q}_2 < \overline{q}_2$.

For the bounds $\underline{q}_1, \overline{q}_1$, the following two assumptions are made.

Assumption 1. The upper bound $\overline{q}_1$ satisfies

$$\overline{q}_1 < \frac{a}{bc}. \tag{7}$$

In addition to being consistent with the instability assumption (6), namely, $0 < q_1 \leq \overline{q}_1 < a/(bc)$, Assumption 1 is also used in the forthcoming design condition (12) where the control gain κ is chosen in accordance with the known bounds on the unknown parameters.

Assumption 2. The difference between the upper and lower bounds $\underline{q}_1 \leq \overline{q}_1$ is smaller than the following unknown constant,

$$\overline{q}_1 - \underline{q}_1 < \sqrt{\frac{q_2 q_1 r_b}{2r_a}}, \tag{8}$$

where r_a, r_b are positive unknown constants, upper and lower bounded, respectively, as

$$r_b < \frac{m}{2c_0^2 q_1}, \tag{9}$$

$$r_a > \frac{2}{q_2}\left(q_1 r_b p^2 + \frac{(q_1 p + q_2)^2 b^2}{2m}\right), \tag{10}$$

the unknown constant m in the bounds (9) and (10) is defined as

$$m = -a + q_1 bc - b\kappa > 0, \tag{11}$$

and the constant κ appearing in (11), and to be used later in control design, is chosen to satisfy

$$\kappa < \min\left\{\frac{(a - \underline{q}_1 bc)\left[\overline{q}_2 + (\overline{q}_1 - \underline{q}_1)\left(\frac{cb(\overline{q}_2 + \overline{q}_1 p)}{a - \overline{q}_1 bc} + p\right)\right]}{-\underline{q}_2 b}, \underline{q}_1 c - \frac{a}{b}\right\}. \tag{12}$$

The purpose of Assumption 2, that is, (8) will become evident in Section 3, with the purpose of (8) becoming evident specifically in inequality (28), whose role is in estimating the exponential decay rate under nominal feedback. This assumption is not required in the BaLSI design and the exact parameter estimation. It is used in the stability analysis by the Lyapunov method in Section 5. If there is no unknown parameter staying with the proximal reflection term $z(1,t)$ in (5), Assumption 2, that is, (8), is not required.

Assumption 2 is difficult to verify a priori for two reasons. First, because the unknown q_1 and q_2 appear in (8), (9), (10), (11). Second, because $\overline{q}_1 - \underline{q}_1$ appears both on the left of (8) and on the right of (12). But Assumption 2 can unquestionably be satisfied for sufficiently small $\overline{q}_1 - \underline{q}_1$. Unfortunately, very small $\overline{q}_1 - \underline{q}_1$ essentially means that the transport speed q_1 is known.

Let us now recap that q_2, which appears in the actuated w-PDE in (3), and is the transport speed in the direction downstream from the input, is arbitrary (positive), whereas the unactuated z-PDE in (2) may have to be nearly perfectly known.

The parameters q_1, q_2 appear in both the PDE as well as the ODE. The structure of the plant and the conditions of the plant parameters come, at least in principle, and as we already indicated above, from writing a wave PDE-ODE coupled model in Riemann coordinates. If the wave PDE's Young modulus were unknown, the transformation into the Riemann variables would contain such an unknown quality, which would render $z(x,t)$ and $w(x,t)$ unmeasurable. We proceed with an adaptive design for the class of systems (2), (3) with the expectation that applications do exist in which the transformation step into (2), (3) is not needed and (z, w) are measurable.

If the original plant were a wave PDE, the ODE (1) would be driven by the wave PDE's boundary state of Neumann type, multiplied by a coefficient associated with the wave PDE's propagation velocity, while the opposite boundary of the wave PDE would be actuated using Neumann actuation with a coefficient associated with the wave PDE's propagation velocity as well. One physical model of this type system is an oil well drilling model,[51] where

$$a = \frac{c_a}{I_B}, \quad b = \frac{I_d}{2I_B}, \quad q_1 = q_2 = \sqrt{\frac{GJ}{I_d}}, \quad \overline{c} = \frac{2}{I_d}, \quad p = 1, \quad c = 2,$$

with I_d being the moment of inertia per unit of length, G the shear modulus, J the geometrical moment of inertia of the drill pipe, c_a the anti-damping coefficient at the bit due to the stick-slip instability, and I_B the moment of inertia of the Bottom Hole Assembly.

3 | NOMINAL CONTROL DESIGN

We introduce the following backstepping transformation

$$\alpha(x,t) = z(x,t), \tag{13}$$

$$\beta(x,t) = w(x,t) - \int_0^x \phi(x,y)w(y,t)dy - \lambda(x)\zeta(t), \tag{14}$$

where

$$\lambda(x;q_1,q_2) = \frac{\kappa}{q_1 p + q_2} e^{\frac{1}{q_2}(a-q_1bc)x}, \tag{15}$$

$$\phi(x,y;q_1,q_2) = \frac{\kappa}{q_2} e^{\frac{1}{q_2}(a-q_1bc)(x-y)} b, \tag{16}$$

and κ is a design parameter, first mentioned in Assumption 2, and to be chosen according to (12).

Writing q_1, q_2 after "; " in $\lambda(x;q_1,q_2)$ and $\phi(x,y;q_1,q_2)$ emphasizes the fact that $\phi(x,y)$, $\lambda(x)$ depend on the unknown parameters q_1, q_2.

By applying the backstepping transformations (13), (14), the plants (1)–(5) is converted to the target system

$$\dot{\zeta}(t) = -m\zeta(t) + b(q_1 p + q_2)\beta(0,t), \tag{17}$$

$$\alpha(0,t) = c_0\zeta(t) - p\beta(0,t), \tag{18}$$

$$\alpha_t(x,t) = -q_1\alpha_x(x,t), \tag{19}$$

$$\beta_t(x,t) = q_2\beta_x(x,t), \tag{20}$$

$$\beta(1,t) = 0, \tag{21}$$

where

$$c_0 = c - p\lambda(0;q_1,q_2). \tag{22}$$

The control input $U(t)$ is chosen as

$$U(t) = -\frac{1}{c}[q_1 z(1,t) - q_2\int_0^1 \phi(1,y;q_1,q_2)w(y,t)dy - q_2\lambda(1;q_1,q_2)\zeta(t)], \tag{23}$$

to ensure (21).

Define

$$\Omega(t) = \|z[t]\|^2 + \|w[t]\|^2 + \zeta(t)^2, \tag{24}$$

and a vector θ containing the two unknown parameters as

$$\theta = (q_1, q_2)^T. \tag{25}$$

Through Lyapunov analysis for the target system (17)–(21), and applying the invertibility of the backstepping transformation, the estimate

$$\Omega(t) \le \Upsilon_\theta \Omega(0) e^{-\lambda_1 t}, \quad t \ge 0, \tag{26}$$

is obtained, where the decay rate λ_1 is

$$\lambda_1 = \min\{\frac{1}{2}(q_1bc - a - b\kappa), \delta q_2, \delta q_1\}, \tag{27}$$

with the analysis parameter $\delta > 0$ selected as

$$\delta \leq \ln\left(\frac{1}{\overline{q}_1 - \underline{q}_1}\sqrt{\frac{q_2 q_1 r_b}{2r_a}}\right), \tag{28}$$

in order to meet the needs of the Lyapunov analysis, which will become evident in Section 5. For the right-hand side of (28) to be positive, we need

$$\frac{1}{(\overline{q}_1 - \underline{q}_1)}\sqrt{\frac{q_2 q_1 r_b}{2r_a}} > 1. \tag{29}$$

This is ensured by Assumption 2. To recap, δ in (28) is only an analysis parameter, which influences the decay rate in (26).

The overshoot coefficient Υ_θ obtained in (26), through the straightforward and omitted Lyapunov analysis, is

$$\Upsilon_\theta = \frac{\xi_2\xi_4}{\xi_1\xi_3}, \tag{30}$$

where the positive constants $\xi_1, \xi_2, \xi_3, \xi_4$ are

$$\xi_1 = \min\left\{\frac{1}{2}r_a, \frac{1}{2}r_b e^{-\delta}, \frac{1}{2}\right\}, \tag{31}$$

$$\xi_2 = \max\left\{\frac{1}{2}r_a e^{\delta}, \frac{1}{2}r_b, \frac{1}{2}\right\}, \tag{32}$$

$$\xi_3 = \frac{1}{\max\left\{3 + \frac{3\kappa^2 b^2}{{q_2}^2}\|\overline{m}\|^2, \frac{3\kappa^2}{(q_2+q_1p)^2}\|\overline{m}\|^2 + 1\right\}}, \tag{33}$$

$$\xi_4 = \max\left\{3 + \frac{3\kappa^2 b^2}{{q_2}^2}\|\overline{n}\|^2, \frac{3\kappa^2}{(q_2 + q_1p)^2}\|\overline{n}\|^2 + 1\right\}, \tag{34}$$

with

$$\overline{m}(x) = e^{\frac{a-q_1bc+b\kappa}{q_2}x}, \quad \overline{n}(x) = e^{\frac{1}{q_2}(a-q_1bc)x}, \tag{35}$$

and with the positive constants r_a, r_b required in Assumption 2 to satisfy (10), (9).

The relations (30)–(35), (9), (10), (28) will be used in the proofs of the main results in Section 5.

We refer to the controller $U(t)$ in (23) as the nominal feedback, which requires the knowledge of the values of the parameters q_1, q_2. The adaptive scheme working with the nominal feedback (23) and guaranteeing exponential regulation is presented in the next section.

4 | REGULATION-TRIGGERED ADAPTIVE CONTROL

The regulation-triggered adaptive control includes a certainty-equivalence controller and a least-squares identifier which is updated at a sequence of time instants.

4.1 | The certainty-equivalence controller

The control action in the interval between two consecutive events is the result of replacing the unknown parameters q_1, q_2 in the nominal control law (23) by their estimates $\hat{q}_1, \hat{q}_2$ at the beginning of the interval, with the estimates $\hat{q}_1, \hat{q}_2$ kept constant during the interval. In other words, the adaptive version of (23) is given by

$$U(t) = -\frac{1}{c}[\hat{q}_1(\tau_i)z(1,t) - \hat{q}_2(\tau_i)\int_0^1 \phi(1,y;\hat{\theta}(\tau_i))w(y,t)dy - \hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i))\zeta(t)], \quad t \in [\tau_i, \tau_{i+1}), \quad i \in Z_+, \tag{36}$$

$$\hat{\theta}(t) = (\hat{q}_1(t), \hat{q}_2(t))^T = (\hat{q}(\tau_i), \hat{q}_2(\tau_i))^T = \hat{\theta}(\tau_i), \quad t \in [\tau_i, \tau_{i+1}), \quad i \in Z_+, \tag{37}$$

where $\{\tau_i \geq 0\}_{i=0}^{\infty}$ is the sequence of time instants, which, along with the estimates $\hat{\theta}(\tau_i)$, is defined next.

4.2 | The event-trigger

The sequence of time instants $\{\tau_i \geq 0\}_{i=0}^{\infty}$ is chosen to satisfy

$$\tau_{i+1} = \min\{\tau_i + T, r_i\}, \quad i \in Z_+, \tag{38}$$

with $\tau_0 = 0$. The constant $T > 0$ is a design parameter with the purpose to avoid a low update frequency and, more importantly, $r_i > \tau_i$ is a time instant determined by an event trigger which is designed next. The trigger was introduced in Reference 47 and is based on the progress of the regulation of the states.

The event trigger sets $r_i > \tau_i$ to be the smallest value of time $t > \tau_i$ for which

$$\Omega(t) = \Upsilon_{\hat{\theta}(\tau_i)}(1 + \overline{a})\Omega(\tau_i), \tag{39}$$

for $\Omega(\tau_i) \neq 0$, where $\Upsilon_{\hat{\theta}(\tau_i)} \geq 1$ is the coefficient defined by (30) with q_1, q_2 replaced by $\hat{q}_1, \hat{q}_2$, the design parameter $\overline{a}$ is positive, and Ω is defined by (24) with the solutions of (1)–(5) under (36). In simple terms, the parameter estimate update is triggered if the plant norm has grown by a certain factor, specially, by $\Upsilon_{\hat{\theta}(\tau_i)}(1 + \overline{a})$. Since $\Upsilon_{\hat{\theta}(\tau_i)}$ is the overshoot coefficient already associated with the system transient in accordance with the estimate (26), the real net growth factor that triggers the update is $1 + \overline{a}$ for any $\overline{a} > 0$ chosen by the user.

If a time $t > \tau_i$ satisfying (39) does not exist, we set $r_i = +\infty$. For the case that $\Omega(\tau_i) = 0$, we set $r_i := \tau_i + T$. Therefore, the event trigger r_i is built as

$$r_i := \inf\{t > \tau_i : \Omega(t) = \Upsilon_{\hat{\theta}(\tau_i)}(1 + \overline{a})\Omega(\tau_i)\}, \ \Omega(\tau_i) \neq 0, \tag{40}$$

$$r_i := \tau_i + T, \quad \Omega(\tau_i) = 0. \tag{41}$$

The following lemma shows that the event-trigger is well-defined and produces an increasing sequence of events.

Lemma 1. *The event-trigger (38), (40), (41) is well defined, that is, $\tau_{i+1} > \tau_i$, for all $i \in Z_+$.*

Proof. If $\Omega(\tau_i) = 0$, it follows from (38), (41) that $\tau_{i+1} = \tau_i + T$. If $\Omega(\tau_i) \neq 0$ and r_i defined in (40) is less than $\tau_i + T$, the dwell time $\tau_{i+1} - \tau_i$ is greater than zero because $\Omega(\tau_{i+1}) = \Upsilon_{\hat{\theta}(\tau_i)}(1 + \overline{a})\Omega(\tau_i) > \Omega(\tau_i)$ and $\Omega(t)$ defined in (24) is a continuous function on $t \in [\tau_i, \tau_{i+1}]$. If $r_i \geq \tau_i + T$ or r_i is infinite, it follows from (38) that $\tau_{i+1} = \tau_i + T$. ∎

The above lemma allows us to define the solution on the interval $[0, \lim_{i\to\infty}(\tau_i))$.

4.3 | Least-squares identifier

The least-squares identifier activated by the trigger defined by (38)–(41) is designed in this subsection. The design idea of the identifier follows from Reference 47. According to the considered dynamic model, by applying integration, formulating a cost function, using Fermat's theorem, a matrix equation is constructed, with an unknown vector of plant parameters, and with the equation's coefficients being the plant states over a time interval. The parameter estimation is then treated as a convex optimization problem with linear equality constraints.

By virtue of (1)–(5), we get for $\tau > 0$ and $n = 1, 2, \dots$ that

$$\begin{aligned}
&\frac{d}{d\tau}\left(\int_0^1 \cos(x\pi n)z(x,\tau)dx + \int_0^1 \cos(x\pi n)w(x,\tau)dx + \frac{1}{b}\zeta(\tau)\right) \\
&= -q_1(-1)^n z(1,\tau) + q_1 z(0,\tau) - q_1\pi n\int_0^1 \sin(x\pi n)z(x,\tau)dx \\
&\quad + q_2(-1)^n w(1,\tau) - q_2 w(0,\tau) + q_2\pi n\int_0^1 \sin(x\pi n)w(x,\tau)dx \\
&\quad + \frac{a}{b}\zeta(\tau) + (q_2 w(0,\tau) - q_1 z(0,\tau)) \\
&= -q_1\pi n\int_0^1 \sin(x\pi n)z(x,\tau)dx + q_2\pi n\int_0^1 \sin(x\pi n)w(x,\tau)dx + (-1)^n\bar{c}U(\tau) + \frac{a}{b}\zeta(\tau),
\end{aligned} \tag{42}$$

where (4) was inserted into (1) to replace $q_1 bc\zeta(t)$ and to yield

$$\frac{d}{d\tau}\zeta(\tau) = a\zeta(\tau) + b(q_2 w(0,\tau) - q_1 z(0,\tau)). \tag{43}$$

Integrating (42) from μ_{i+1} to t, yields

$$f_n(t,\mu_{i+1}) = q_1 g_{n,1}(t,\mu_{i+1}) + q_2 g_{n,2}(t,\mu_{i+1}), \tag{44}$$

where

$$\begin{aligned}
f_n(t,\mu_{i+1}) = &\left(\int_0^1 \cos(x\pi n)z(x,t)dx + \int_0^1 \cos(x\pi n)w(x,t)dx + \frac{1}{b}\zeta(t)\right) \\
&- \left(\int_0^1 \cos(x\pi n)z(x,\mu_{i+1})dx + \int_0^1 \cos(x\pi n)w(x,\mu_{i+1})dx + \frac{1}{b}\zeta(\mu_{i+1})\right) \\
&- \int_{\mu_{i+1}}^t \left((-1)^n\bar{c}U(\tau) + \frac{a}{b}\zeta(\tau)\right)d\tau,
\end{aligned}$$

$$g_{n,1}(t,\mu_{i+1}) = -\int_{\mu_{i+1}}^t \pi n\int_0^1 \sin(x\pi n)z(x,\tau)dxd\tau, \tag{45}$$

$$g_{n,2}(t,\mu_{i+1}) = \int_{\mu_{i+1}}^t \pi n\int_0^1 \sin(x\pi n)w(x,\tau)dxd\tau, \tag{46}$$

for $n = 1, 2, \dots$. The time μ_{i+1} introduced in Reference 44 is

$$\mu_{i+1} := \min\{\tau_f : f \in \{0, \dots, i\}, \tau_f \geq \tau_{i+1} - \tilde{N}T\}, \tag{47}$$

where the positive integer $\tilde{N} \geq 1$ is a design parameter. In practice, a larger $\tilde{N}$ can reduce the effect of measurement noise on the precision of estimation, with a cost of larger computation.[44]

Equation (44) is written as

$$f_n(t,\mu_{i+1}) = \eta_n(t,\mu_{i+1})\theta, \tag{48}$$

where

$$\eta_n(t,\mu_{i+1}) = [g_{n,1}(t,\mu_{i+1}), g_{n,2}(t,\mu_{i+1})], \tag{49}$$

and θ is defined in (25). Define the function $h_{i,n} : \mathbb{R}^2 \to R_+$ by the formula

$$h_{i,n}(\ell) = \int_{\mu_{i+1}}^{\tau_{i+1}} (f_n(t, \mu_{i+1}) - \eta_n(t, \mu_{i+1})\ell)^2 dt, \tag{50}$$

for $n = 1, 2, \ldots, \ell = [\ell_1, \ell_2]^T, i \in Z_+$.

According to (48), the function $h_{i,n}(\ell)$ (50) has a global minimum $h_{i,n}(\theta) = 0$. We get from Fermat's theorem (vanishing gradient at extrema) that the following equations hold for every $i \in Z_+$ and $n = 1, 2, \ldots$:

$$H_{n,1}(\mu_{i+1}, \tau_{i+1}) = q_1 Q_{n,1}(\mu_{i+1}, \tau_{i+1}) + q_2 Q_{n,2}(\mu_{i+1}, \tau_{i+1}), \tag{51}$$

$$H_{n,2}(\mu_{i+1}, \tau_{i+1}) = q_1 Q_{n,2}(\mu_{i+1}, \tau_{i+1}) + q_2 Q_{n,3}(\mu_{i+1}, \tau_{i+1}), \tag{52}$$

where

$$H_{n,1}(\mu_{i+1}, \tau_{i+1}) = \int_{\mu_{i+1}}^{\tau_{i+1}} g_{n,1}(t, \mu_{i+1}) f_n(t, \mu_{i+1}) dt, \tag{53}$$

$$H_{n,2}(\mu_{i+1}, \tau_{i+1}) = \int_{\mu_{i+1}}^{\tau_{i+1}} g_{n,2}(t, \mu_{i+1}) f_n(t, \mu_{i+1}) dt, \tag{54}$$

$$Q_{n,1}(\mu_{i+1}, \tau_{i+1}) = \int_{\mu_{i+1}}^{\tau_{i+1}} g_{n,1}(t, \mu_{i+1})^2 dt, \tag{55}$$

$$Q_{n,2}(\mu_{i+1}, \tau_{i+1}) = \int_{\mu_{i+1}}^{\tau_{i+1}} g_{n,1}(t, \mu_{i+1}) g_{n,2}(t, \mu_{i+1}) dt, \tag{56}$$

$$Q_{n,3}(\mu_{i+1}, \tau_{i+1}) = \int_{\mu_{i+1}}^{\tau_{i+1}} g_{n,2}(t, \mu_{i+1})^2 dt. \tag{57}$$

Indeed, (51), (52) are obtained by differentiating the functions $h_{i,n}(\ell)$ defined by (50) with respect to ℓ_1, ℓ_2, respectively, and evaluating the derivatives at the position of the global minimum $(\ell_1, \ell_2) = (q_1, q_2)$. Equations (51) and (52) are organized as

$$Z_n(\mu_{i+1}, \tau_{i+1}) = G_n(\mu_{i+1}, \tau_{i+1})\theta, \tag{58}$$

where

$$Z_n(\mu_{i+1}, \tau_{i+1}) = [H_{n,1}(\mu_{i+1}, \tau_{i+1}), H_{n,2}(\mu_{i+1}, \tau_{i+1})]^T, \tag{59}$$

$$G_n(\mu_{i+1}, \tau_{i+1}) = \begin{bmatrix} Q_{n,1}(\mu_{i+1}, \tau_{i+1}) & Q_{n,2}(\mu_{i+1}, \tau_{i+1}) \\ Q_{n,2}(\mu_{i+1}, \tau_{i+1}) & Q_{n,3}(\mu_{i+1}, \tau_{i+1}) \end{bmatrix}. \tag{60}$$

The parameter update law is defined as

$$\hat{\theta}(\tau_{i+1}) = \text{argmin}\{|\ell - \hat{\theta}(\tau_i)|^2 : \ell \in \Theta,\ Z_n(\mu_{i+1}, \tau_{i+1}) = G_n(\mu_{i+1}, \tau_{i+1})\ell,\ n = 1, 2, \ldots\}, \tag{61}$$

where $\Theta = \{\ell \in \mathbb{R}^2 : \underline{q}_1 \le \ell_1 \le \overline{q}_1, \underline{q}_2 \le \ell_2 \le \overline{q}_2\}$. The estimates are updated at τ_{i+1}, that is, $\hat{\theta}(\tau_{i+1}) = (\hat{q}_1(\tau_{i+1}), \hat{q}_2(\tau_{i+1}))^T$, using the plant states over the time interval $[\mu_{i+1}, \tau_{i+1}]$, where the length of the data acquisition can be adjusted by $\tilde{N}$ in (47). The initial values of the estimates $\hat{q}_1(0), \hat{q}_2(0)$ are chosen as $\hat{q}_1(0) = \underline{q}_1, \hat{q}_2(0) = \underline{q}_2$, making $\hat{q}_1(\tau_i) \le q_1$ and $\hat{q}_2(\tau_i) \le q_2$, which will be seen more clearly later. If a more robust identifer with respect to random measurement noise is required, the identifer can be designed in a double integral form as in Reference 44.

For the solution notion, according to definition A.5 in Reference 52, we give the following weak solution definition.

Definition 1. Consider the system

$$\mathcal{R}_t + \Lambda(x)\mathcal{R}_x + M(x)\mathcal{R} = 0, \quad t \in [0,\infty), \quad x \in [0,L], \tag{62}$$

$$\begin{pmatrix} \mathcal{R}^+(t,0) \\ \mathcal{R}^-(t,L) \end{pmatrix} = K \begin{pmatrix} R^+(t,L) \\ R^-(t,0) \end{pmatrix} + \begin{pmatrix} N^+ \\ N^- \end{pmatrix} X + \int_0^L \begin{pmatrix} F^+(x) \\ F^-(x) \end{pmatrix} \mathcal{R} dx \tag{63}$$

$$\frac{dX}{dt} = E^+\mathcal{R}^+(t,L) + E^-\mathcal{R}^+(t,0) + E_0X, \quad X \in \mathbb{R}^p, \tag{64}$$

$$\mathcal{R}(0,x) = \mathcal{R}_0(x), X(0) = X_0, \tag{65}$$

where $\mathcal{R} : [0,+\infty) \times [0,L] \to \mathbb{R}^n, M : [0,L] \to \mathcal{M}_{n,n}(\mathbb{R})$, the symbol $\mathcal{M}_{n,n}(\mathbb{R})$, as usual, denotes the set of $n \times n$ real matrices, $F^+ : [0,L] \to \mathcal{M}_{m,n}(\mathbb{R})$, $F^- : [0,L] \to \mathcal{M}_{n-m,n}(\mathbb{R})$, and $\Lambda(x) \triangleq \text{diag}\{\Lambda^+(x), \Lambda^-(x)\}$ such that

$$\Lambda^+(x) \triangleq \text{diag}\{\lambda_1(x), \dots, \lambda_m(x)\}, \tag{66}$$

$$\Lambda^-(x) \triangleq -\text{diag}\{\lambda_{m+1}(x), \dots, \lambda_n(x)\}, \tag{67}$$

with $\lambda_i(x) > 0, \forall x \in [0,L]$, and where

$$K \triangleq \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}, K_{00} \in \mathcal{M}_{m,m}(\mathbb{R}), K_{01} \in \mathcal{M}_{m,n-m}(\mathbb{R}), \tag{68}$$

$$K_{10} \in \mathcal{M}_{n-m,m}(\mathbb{R}), \quad K_{11} \in M_{n-m,n-m}(\mathbb{R}), \tag{69}$$

$$N^+ \in \mathbb{R}^{m\times p}, \quad N^- \in \mathbb{R}^{(n-m)\times p}, \tag{70}$$

$$E^+ \in \mathbb{R}^{p\times m}, \quad E^- \in \mathbb{R}^{p\times(n-m)}, \quad E_0 \in \mathbb{R}^{p\times p}. \tag{71}$$

A solution $\mathcal{R}$: $(0,+\infty) \times (0,L) \to \mathbb{R}^n, X : (0,\infty) \to \mathbb{R}^p$ of the system (62)–(65) is a map $\mathcal{R} \in C^0([0,+\infty); L^2(0,1); \mathbb{R}^n)$, $X \in C^0([0,+\infty); \mathbb{R}^p)$ satisfying (65) such that, for every $T > 0$, every $\psi \in C^1([0,T] \times [0,L]; \mathbb{R}^n)$, and every $\eta \in C^1([0,T]; \mathbb{R}^p)$ satisfying

$$\begin{pmatrix} \psi^+(t,L) \\ \psi^-(t,0) \end{pmatrix} = \begin{pmatrix} \Lambda^+(L)^{-1}K_{00}^T\Lambda^+(0) & \Lambda^+(L)^{-1}K_{10}^T\Lambda^-(L) \\ \Lambda^-(0)^{-1}K_{01}^T\Lambda^+(0) & \Lambda^-(0)^{-1}K_{11}^T\Lambda^-(L) \end{pmatrix} \times \begin{pmatrix} \psi^+(t,0) \\ \psi^-(t,L) \end{pmatrix} + \begin{pmatrix} \Lambda^+(L)E^{+T} \\ \Lambda^+(0)E^{-T} \end{pmatrix} \eta, \tag{72}$$

we have

$$\begin{aligned} &\int_0^L \psi(T,x)^T\mathcal{R}(T,x)dx - \int_0^L \psi(0,x)^T\mathcal{R}_0(x) + \eta^T(T)X(T) - \eta^T(0)X(0) \\ &= \int_0^T \int_0^L [\psi_t^T + \psi_x^T\Lambda + \psi^T(\Lambda_x - M) + \psi^{-T}(t,L)\Lambda^-(L)F^- + \psi^{+T}(t,0)\Lambda^+(0)F^+]\mathcal{R}dxdt \\ &\quad + \int_0^T [\eta_t^T + \eta^TE_0 + \psi^{-T}(t,L)\Lambda^-(L)N^- + \psi^{+T}(t,0)\Lambda^+(0)N^+]Xdt. \end{aligned} \tag{73}$$

Proposition 1. *For every $(z_0, w_0)^T \in L^2((0,1); \mathbb{R}^2)$, $\zeta_0 \in \mathbb{R}$ and $\hat{\theta}_0 \in \Theta$, the initial boundary value problem (1)–(5) with (36), (37), (38), (40), (41), (47), (61) and initial conditions $w[0] = w_0$, $z[0] = z_0$, $\zeta(0) = \zeta_0$, $\hat{\theta}(0) = \hat{\theta}_0$, has a unique (weak) solution $((z,w)^T, \zeta) \in C^0([0, \lim_{k\to\infty}(\tau_k)); L^2(0,1); \mathbb{R}^2) \times C^0([0, \lim_{k\to\infty}(\tau_k)); \mathbb{R})$.*

Proof. The proof is shown in Appendix A. ■

The flow chart of the mechanism of the regulation-triggered adaptive controller is shown in Figure 1, and some system properties are given in the following lemmas. In the rest of this paper, when we say that $z[t]$, $w[t]$ are equal to zero for $x \in [0,1], t \in [\mu_{i+1}, \tau_{i+1}]$, or not identically zero on the same domain, we mean "except possibly for finitely many discontinuities of the functions $w[t], z[t]$." These discontinuities are isolated curves in the rectangle $[0,1] \times [\mu_{i+1}, \tau_{i+1}]$.

Lemma 2. *The sufficient and necessary condition of* $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) = 0$ *(or* $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) = 0$*) for* $n = 1, 2, \ldots$ *is* $z[t] = 0$ *(or* $w[t] = 0$*) on* $t \in [\mu_{i+1}, \tau_{i+1}]$.

Proof. Necessity: If $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) = 0$ for $n = 1, 2, \ldots$, then the definition (55) in conjunction with continuity of $g_{n,1}(t, \mu_{i+1})$ for $t \in [\mu_{i+1}, \tau_{i+1}]$ (a consequence of definition (45) and the fact that $z \in C^0([\mu_{i+1}, \tau_{i+1}]; L^2(0,1))$) implies

$$g_{n,1}(t, \mu_{i+1}) = 0, \quad t \in [\mu_{i+1}, \tau_{i+1}]. \tag{74}$$

According to the definition (45) and continuity of the mapping $\tau \to \int_0^1 \sin(x\pi n) z[\tau] dx$ (a consequence of the fact that $z \in C^0([\mu_{i+1}, \tau_{i+1}]; L^2(0,1))$, (74) implies

$$\int_0^1 \sin(x\pi n) z(x, \tau) dx = 0, \quad \tau \in [\mu_{i+1}, \tau_{i+1}], \tag{75}$$

for $n = 1, 2, \ldots$. Since the set $\{\sqrt{2}\sin(n\pi x) : n = 1, 2, \ldots\}$ is an orthonormal basis of $L^2(0,1)$, we have $z[t] = 0$ for $t \in [\mu_{i+1}, \tau_{i+1}]$.

Similarly, if $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) = 0$ for $n = 1, 2, \ldots$, then $w[t] = 0$ on $t \in [\mu_{i+1}, \tau_{i+1}]$, recalling the definitions (57), (46), and the fact that $w \in C^0([\mu_{i+1}, \tau_{i+1}]; L^2(0,1))$, and the set $\{\sqrt{2}\sin(n\pi x) : n = 1, 2, \ldots\}$ being an orthonormal basis of $L^2(0,1)$.

Sufficiency: If $z[t] = 0$ on $t \in [\mu_{i+1}, \tau_{i+1}]$ (or $w[t] = 0$ on $t \in [\mu_{i+1}, \tau_{i+1}]$), then $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) = 0$ (or $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) = 0$) for $n = 1, 2, \ldots$ is obtained directly, according to (45), (55) and (46), (57).

The proof of Lemma 2 is complete. ■

Lemma 3. *For the adaptive estimates defined by (61) based on the data in the interval* $t \in [\mu_{i+1}, \tau_{i+1}]$*, the following statements hold:*

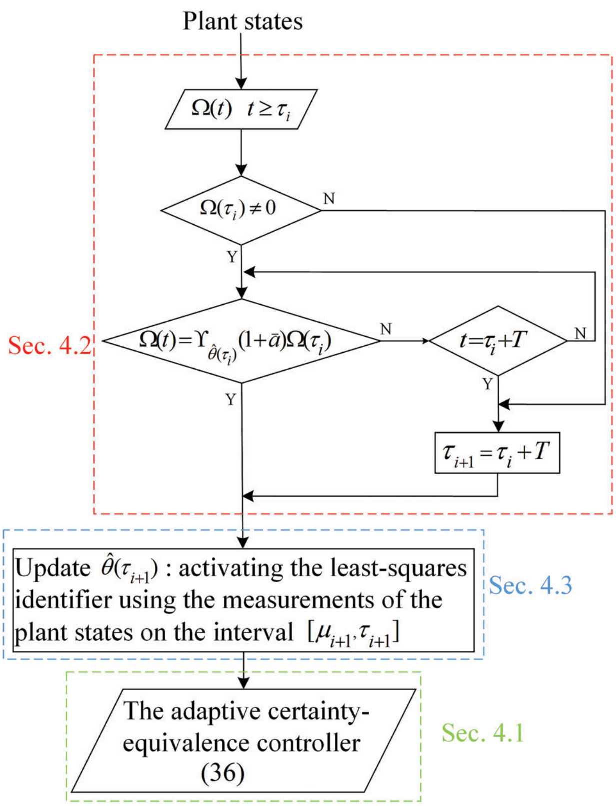

FIGURE 1 The adaptive certainty-equivalence control scheme with regulation-triggered batch least-squares identification

(1) If $z[t]$ is not identically zero and $w[t]$ is identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$, then $\hat{q}_1(\tau_{i+1}) = q_1$, $\hat{q}_2(\tau_{i+1}) = \hat{q}_2(\tau_i)$;
(2) If $w[t]$ is not identically zero and $z[t]$ is identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$, then $\hat{q}_1(\tau_{i+1}) = \hat{q}_1(\tau_i)$, $\hat{q}_2(\tau_{i+1}) = q_2$;
(3) If $w[t]$, $z[t]$ are identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$, then $\hat{q}_1(\tau_{i+1}) = \hat{q}_1(\tau_i)$, $\hat{q}_2(\tau_{i+1}) = \hat{q}_2(\tau_i)$;
(4) If both $w[t]$ and $z[t]$ are not identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$, then $\hat{q}_1(\tau_{i+1}) = q_1$, $\hat{q}_2(\tau_{i+1}) = q_2$.
Moreover, if $\hat{q}_1(\tau_i) = q_1$ (or $\hat{q}_2(\tau_i) = q_2$) for certain $i \in Z_+$, then $\hat{q}_1(t) = q_1$ (or $\hat{q}_2(t) = q_2$) for all $t \in [\tau_i, \lim_{k\to\infty}(\tau_k))$.

Proof. Define the following set

$$S_i := \{\ell \in \Theta : Z_n(\mu_{i+1}, \tau_{i+1}) = G_n(\mu_{i+1}, \tau_{i+1})\ell, \quad n = 1, 2, \dots \}. \tag{76}$$

If S_i is a singleton then it is nothing else but the least-squares estimate of the unknown vector of parameters (q_1, q_2) on the interval $[\mu_{i+1}, \tau_{i+1}]$.

(1) Because $z[t]$ is not identically zero and $w[t]$ is identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$, there exists $n \in \mathbb{N}$ such that $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) \neq 0$ recalling Lemma 2. Define the index set I to be the set of all $n \in \mathbb{N}$ with $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) \neq 0$. According to (46) and $w[t]$ being identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$, we know that $g_{n,2}(t, \mu_{i+1}) = 0$ on $t \in [\mu_{i+1}, \tau_{i+1}]$ for $n = 1, 2, \dots$. It follows that $Q_{n,2}(\mu_{i+1}, \tau_{i+1}) = 0$, $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) = 0$, $H_{n,2}(\mu_{i+1}, \tau_{i+1}) = 0$ for $n = 1, 2, \dots$ recalling (56), (57), and (54). Then (76) implies $S_i = \{(\ell_1, \ell_2) \in \Theta : \ell_1 = \frac{H_{n,1}(\mu_{i+1}, \tau_{i+1})}{Q_{n,1}(\mu_{i+1}, \tau_{i+1})}, n \in I\}$ recalling (59), (60). Because $(q_1, q_2) \in S_i$ according to (58), it follows that $S_i = \{(q_1, \ell_2) \in \Theta : \underline{q}_2 \leq \ell_2 \leq \overline{q}_2\}$. Therefore (61) shows that $\hat{q}_1(\tau_{i+1}) = q_1$ and $\hat{q}_2(\tau_{i+1}) = \hat{q}_2(\tau_i)$.

(2) The proof of (2) is very similar to the proof of 1), and thus it is omitted.

(3) Because $w[t], z[t]$ are identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$, then $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) = 0$, $Q_{n,2}(\mu_{i+1}, \tau_{i+1}) = 0$, $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) = 0$, $H_{n,1}(\mu_{i+1}, \tau_{i+1}) = 0$, $H_{n,2}(\mu_{i+1}, \tau_{i+1}) = 0$ for $n = 1, 2, \dots$ according to (45), (46), (53)–(57). It follows that $S_i = \Theta$, and then (61) shows that $\hat{q}_1(\tau_{i+1}) = \hat{q}_1(\tau_i)$, $\hat{q}_2(\tau_{i+1}) = \hat{q}_2(\tau_i)$.

(4) Because $w[t]$ (or $z[t]$) are not identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$, there exists $n \in \mathbb{N}$ such that $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) \neq 0$ (or $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) \neq 0$) recalling Lemma 2. Define the index set I_1 to be the set of all $n \in \mathbb{N}$ with $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) \neq 0$ and define the index set I_2 to be the set of all $n \in \mathbb{N}$ with $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) \neq 0$. Denote the elements in I_1 as $n_1 \in \mathbb{N}$ and those in I_2 as $n_2 \in \mathbb{N}$, that is, $Q_{n_1,1}(\mu_{i+1}, \tau_{i+1}) \neq 0$, $Q_{n_2,3}(\mu_{i+1}, \tau_{i+1}) \neq 0$.

From (76), recalling (59)–(60), we obtain

$$S_i \subseteq \overline{S}_{ai} := \{(\ell_1, \ell_2) \in \Theta : \ell_1 = \frac{H_{n_1,1}(\mu_{i+1}, \tau_{i+1})}{Q_{n_1,1}(\mu_{i+1}, \tau_{i+1})} - \ell_2 \frac{Q_{n_1,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_1,1}(\mu_{i+1}, \tau_{i+1})}, n_1 \in I_1\}, \tag{77}$$

$$S_i \subseteq \overline{S}_{bi} := \{(\ell_1, \ell_2) \in \Theta : \ell_2 = \frac{H_{n_2,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_2,3}(\mu_{i+1}, \tau_{i+1})} - \ell_1 \frac{Q_{n_2,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_2,3}(\mu_{i+1}, \tau_{i+1})}, n_2 \in I_2\}. \tag{78}$$

We next prove by contradiction that $S_i = \{(q_1, q_2)\}$. Suppose that on the contrary $S_i \neq \{(q_1, q_2)\}$, that is, S_i defined by (76) is not a singleton, which implies the sets $\overline{S}_{ai}$, $\overline{S}_{bi}$ defined by (77), (78) are not singletons (because either of $\overline{S}_{ai}$, $\overline{S}_{bi}$ being a singleton implies that S_i is a singleton). It follows that there exist constants $\overline{\lambda} \in \mathbb{R}$, $\overline{\lambda}_1 \in \mathbb{R}$ such that

$$\frac{Q_{n_1,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_1,1}(\mu_{i+1}, \tau_{i+1})} = \overline{\lambda}_1, \quad n_1 \in I_1, \tag{79}$$

$$\frac{Q_{n_2,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_2,3}(\mu_{i+1}, \tau_{i+1})} = \overline{\lambda}, \quad n_2 \in I_2, \tag{80}$$

because if there were two different indices $k_1, k_2 \in I_2$ with $\frac{Q_{k_1,2}(\mu_{i+1}, \tau_{i+1})}{Q_{k_1,3}(\mu_{i+1}, \tau_{i+1})} \neq \frac{Q_{k_2,2}(\mu_{i+1}, \tau_{i+1})}{Q_{k_2,3}(\mu_{i+1}, \tau_{i+1})}$, then the set $\overline{S}_{bi}$ defined by (78) would be a singleton, and the same would be the case with $\overline{S}_{ai}$ defined by (77) if there were two different indices $\overline{k}_1, \overline{k}_2 \in I_1$ with $\frac{Q_{\overline{k}_1,2}(\mu_{i+1}, \tau_{i+1})}{Q_{\overline{k}_1,1}(\mu_{i+1}, \tau_{i+1})} \neq \frac{Q_{\overline{k}_2,2}(\mu_{i+1}, \tau_{i+1})}{Q_{\overline{k}_2,1}(\mu_{i+1}, \tau_{i+1})}$.

Moreover, since S_i is not a singleton, definition (76) implies

$$Q_{n,2}(\mu_{i+1}, \tau_{i+1})^2 = Q_{n,1}(\mu_{i+1}, \tau_{i+1}) Q_{n,3}(\mu_{i+1}, \tau_{i+1}), \tag{81}$$

for all $n \in I_1 \cup I_2$ ((81) naturally holds for $n \notin I_1 \cup I_2$ if $\complement_{\mathbb{N}}\{I_1 \cup I_2\} \neq \emptyset$, because both sides of (81) are zero) by recalling (60) (because if (81) does not hold , it follows from (60) that there exists $n \in I_1 \cup I_2$ such that $\det(G_n(\mu_{i+1}, \tau_{i+1})) \neq 0$, which implies S_i defined by (76) is a singleton: a contradiction). According to (81), (55)–(57), and the fact that the Cauchy–Schwarz inequality holds as equality only when two functions are linearly dependent, we obtain the existence of constants $\hat{\lambda}_{n_1} \in \mathbb{R}$, $\check{\lambda}_{n_2} \in \mathbb{R}$ such that

$$g_{n_1,2}(t, \mu_{i+1}) = \hat{\lambda}_{n_1} g_{n_1,1}(t, \mu_{i+1}), \quad n_1 \in I_1, \tag{82}$$

$$g_{n_2,1}(t, \mu_{i+1}) = \check{\lambda}_{n_2} g_{n_2,2}(t, \mu_{i+1}), \quad n_2 \in I_2, \tag{83}$$

for $t \in [\mu_{i+1}, \tau_{i+1}]$ (notice that $g_{n_1,1}(t, \mu_{i+1})$ and $g_{n_2,2}(t, \mu_{i+1})$ are not identically zero on $t \in [\mu_{i+1}, t_{i+1}]$ because of $Q_{n_1,1}(\mu_{i+1}, \tau_{i+1}) \neq 0$ and $Q_{n_2,3}(\mu_{i+1}, \tau_{i+1}) \neq 0$). Recalling (79), (80), we obtain from (55)–(57) and (82), (83) that

$$g_{n_1,2}(t, \mu_{i+1}) = \overline{\lambda}_1 g_{n_1,1}(t, \mu_{i+1}), \quad n_1 \in I_1, \tag{84}$$

$$g_{n_2,1}(t, \mu_{i+1}) = \overline{\lambda} g_{n_2,2}(t, \mu_{i+1}), \quad n_2 \in I_2, \tag{85}$$

for $t \in [\mu_{i+1}, \tau_{i+1}]$. Equations (84) and (85) holding is a necessary condition of the hypothesis that S_i is not a singleton. The remaining proof of Case 4 is divided into the following three Claims.

Claim 1. If S_i is not a singleton, then $\overline{\lambda} \neq 0, \overline{\lambda}_1 \neq 0$ and $\overline{\lambda} = \frac{1}{\overline{\lambda}_1}$ in (84) and (85).

Proof. The proof is shown in Appendix B. ∎

Claim 2. Equations (84) and (85) ($\overline{\lambda} \neq 0, \overline{\lambda}_1 \neq 0$ and $\overline{\lambda} = \frac{1}{\overline{\lambda}_1}$) hold if and only if $z[t] + \overline{\lambda} w[t] = 0$ ($\overline{\lambda} \neq 0$) for $t \in [\mu_{i+1}, \tau_{i+1}]$.

Proof. The proof is shown in Appendix C. ∎

Claim 3. The function $z[t] + \overline{\lambda} w[t]$ ($\overline{\lambda} \neq 0$) is not identically zero for $t \in [\mu_{i+1}, \tau_{i+1}]$.

Proof. The proof is shown in Appendix D. ∎

Recalling Claims 1–3, we know that (84) and (85), which is a necessary condition of the hypothesis below (78) that S_i not be a singleton, does not hold. Consequently, S_i is a singleton, that is, $S_i = \{(q_1, q_2)\}$. Therefore (61) shows that $\hat{q}_1(\tau_{i+1}) = q_1$, $\hat{q}_2(\tau_{i+1}) = q_2$. The proof of Case 4 is complete.

If $\hat{q}_1(\tau_i) = q_1$ (or $\hat{q}_2(\tau_i) = q_2$) for certain $i \in Z_+$, recalling (61) and the analysis in the above four cases, we have that $\hat{q}_1(t_{i+1}) = q_1$ (or $\hat{q}_2(t_{i+1}) = q_2$). Repeating the above process, we then have that $\hat{q}_1(t) = q_1$ (or $\hat{q}_2(t) = q_2$) for all $t \in [\tau_i, \lim_{k\to\infty}(\tau_k))$.

The proof of Lemma 3 is complete. ∎

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Theorem 1. *With arbitrary initial data $(z_0, w_0)^T \in L^2((0,1); \mathbb{R}^2)$, $\zeta_0 \in \mathbb{R}$, and $\hat{\theta}_0 = (\underline{q}_1, \underline{q}_2)^T$, for the plant (1)–(5) under the adaptive certainty-equivalence boundary controller (36) where the regulation-triggered BaLSI is defined by (37), (61) with (38), (40), (41), (47), the closed-loop system satisfies the following properties:*

(1) The Zeno phenomenon does not occur, that is,

$$\lim_{i\to\infty} \tau_i = +\infty, \tag{86}$$

and the closed-loop system is well-posed.

(2) If the finite time convergence of parameter estimates to the true values does not occur, $\Omega(t)$ reaches zero in finite time $\frac{1}{q_1}$, that is, $\Omega(t) \equiv 0$ on $t \in [\frac{1}{q_1}, \infty)$.

(3) If the parameter estimates converge to the true values in finite time, there exist positive constants $M_{\theta,\hat{\theta}(0)}$, λ_1 such that

$$\Omega(t) \leq M_{\theta,\hat{\theta}(0)}\Omega(0)e^{-\lambda_1 t}, \quad t \geq 0, \tag{87}$$

where $\Omega(t)$ is given in (24), and $M_{\theta,\hat{\theta}(0)}$ is a family of constants parameterized by positive constants $q_1, q_2, \hat{q}_1(0), \hat{q}_2(0)$. The decay rate λ_1 is the same as the nominal control result in (26).

Proof. First, we propose the following claim about the sufficient and necessary condition of the finite time convergence of parameter estimates to the true values.

Claim 4. When $\hat{q}_1(0) \neq q_1$ (or $\hat{q}_2(0) \neq q_2$), the estimate $\hat{q}_1(t)$ (or $\hat{q}_2(t)$) reaches the actual value q_1 (or q_2) in finite time if and only if $z[t]$ (or $w[t]$) is not identically zero on $t = [0, \lim_{i\to\infty}(\tau_i))$.

Proof. The proof is shown in Appendix E. ∎

(1) Now we prove the first of the three portions of the theorem. First, if the estimates $\hat{q}_1(t), \hat{q}_2(t)$ reach the true values in finite time τ_ϵ, we have that $\tau_j = \tau_\epsilon + (j-\epsilon)T, j \in Z_+, j > \epsilon$. The proof of this is shown next. We prove by induction that $\tau_{i+1} = \tau_i + T$ for $i \geq \epsilon$. Let $i \geq \epsilon$ be an integer. Notice that (23) holds for all $t \in [\tau_i, \tau_{i+1})$. Assume that $\Omega(\tau_i) \neq 0$. By virtue of (26) and since (23) holds, we have

$$\Omega(t) \leq \Upsilon_{\hat{\theta}(\tau_i)}\Omega(\tau_i), \tag{88}$$

for all $t \in [\tau_i, \tau_{i+1})$. It follows that

$$\Omega(t) \leq \Upsilon_{\hat{\theta}(\tau_i)}\Omega(\tau_i) < \Upsilon_{\hat{\theta}(\tau_i)}(1+\overline{a})\Omega(\tau_i), \tag{89}$$

for $t \in [\tau_i, \tau_{i+1})$ where $\overline{a}$ is positive. Therefore, we get from (38), (40) that $\tau_{i+1} = \tau_i + T$ for $i \geq \epsilon$. The same conclusion follows from (38) and (41) if $\Omega(\tau_i) = 0$. Therefore, $\lim_{i\to\infty}(\tau_i) = +\infty$.

If the finite time convergence of the parameter estimates to the true values is not achieved, the proof is divided into the three cases.

Case 1: We suppose that the estimate $\hat{q}_2(t)$ does not reach q_2 in finite time but $\hat{q}_1(t)$ does reach q_1 in finite time. The fact that $\hat{q}_2(t)$ does not reach q_2 in finite time implies $w[t] \equiv 0$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$ according to Claim 4, and $\hat{q}_2(t) = \hat{q}_2(0) \neq q_2$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$ according to Lemma 3. The fact that $\hat{q}_1(t)$ reaches q_1 in finite time implies $\tilde{q}_1(t) \equiv 0$ after a certain τ_f.

Inserting (36) into (5), we obtain

$$q_2 w(1,t) = \hat{q}_2(0)\int_0^1 \phi(1,y;\hat{q}_1(t),\hat{q}_2(0))w(y,t)dy + \hat{q}_2(0)\lambda(1;\hat{q}_1(t),\hat{q}_2(0))\zeta(t) + \tilde{q}_1(t)z(1,t). \tag{90}$$

Considering $w[t] \equiv 0$ and $\tilde{q}_1(t) \equiv 0$ on $t \in [\tau_f, \lim_{i\to\infty}(\tau_i))$, we obtain from (90) that $\zeta(t) \equiv 0$ on $t \in [\tau_f, \lim_{i\to\infty}(\tau_i))$ because of $\hat{q}_2(0) \neq 0$ and $\lambda(1;\hat{q}_1(t),\hat{q}_2(0)) \neq 0$. Recalling (1), considering $w[t] \equiv 0$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$ and $\zeta(t) \equiv 0$ on $t \in [\tau_f, \lim_{i\to\infty}(\tau_i))$, it further follows that $\zeta(0) = 0$, that is, $\zeta(t) \equiv 0$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$, which means that $z(0,t) \equiv 0$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$. It follows that $\Omega(t)$ defined in (24) is nonincreasing on $t \in [0, \frac{1}{q_1}]$. Therefore, we have that $\lim_{i\to\infty}(\tau_i) > \frac{1}{q_1}$ according to the definition of triggering times (38), (40), (41). By virtue of (2), (4) and $z(0,t) \equiv 0$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$, we have that $z[t] \equiv 0$ for $t \in [\frac{1}{q_1}, \lim_{i\to\infty}(\tau_i))$. If $z[t]$ is identically zero on $t \in [0, \frac{1}{q_1}]$, then $\hat{q}_1(0) = q_1$ (because $z[t] \equiv 0$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$, and $\hat{q}_1(t)$ reaches q_1 in finite time only when the initial estimate is the true value according to Claim 4). If $z[t]$ is not identically zero in $t \in [0, \frac{1}{q_1})$, it implies that the initial condition $z(x,0)$ is not identically zero for $x \in (0,1]$, moreover, that τ_f must be less than $\frac{1}{q_1}$ and the function $z(x,0)$ is not identically zero on the interval $(0, 1-q_1\tau_f]$ for x (otherwise $w[t]$ is not identically zero according to (90): a contradiction). The state $z(x,t)$ propagates from its initial condition $z(x,0)$, which is possibly not identically zero only on $(0, 1-q_1\tau_f]$, toward the boundary $x=1$ and finally vanishes not latter than $t = \frac{1}{q_1}$ ($z(1,t) = 0$ for $t \in [0,\tau_f)$ and the nonzero values of $z(1,t)$ on $t \in [\tau_f, \frac{1}{q_1}]$ are eliminated by $\tilde{q}_1(t) = 0$ in (90)). Together with $w[t] \equiv 0$, $\zeta(t) \equiv 0$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$, we conclude that $\Omega(t)$ is nonincreasing in $t \in [0, \frac{1}{q_1}]$, and $\Omega(t) \equiv 0$ for $t \in [\frac{1}{q_1}, \lim_{i\to\infty}(\tau_i))$. Therefore, $\tau_j = jT, j \in Z_+$, according to the definition of triggering times (38), (40), (41).

Case 2: We suppose that the estimate $\hat{q}_1(t)$ does not reach q_1 in finite time but $\hat{q}_2(t)$ does reach q_2 in finite time. The fact that $\hat{q}_1(t)$ does not reach q_1 in finite time implies that $z(x,t) \equiv 0$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$ according to Claim 4, and $\hat{q}_1(t) = \hat{q}_1(0) \neq q_1$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$ according to Lemma 3. Recalling (36), then (1)–(5) become

$$\zeta(t) = e^{[(a-q_1bc)+b(q_2+q_1p)\frac{c}{p}]t}\zeta(0), \tag{91}$$

$$w_t(x,t) = q_2 w_x(x,t), \tag{92}$$

$$w(0,t) = \frac{c}{p}\zeta(t), \tag{93}$$

$$w(1,t) = \frac{\hat{q}_2(t)}{q_2}\left[\int_0^1 \phi(1,y;\hat{q}_1(0),\hat{q}_2(t))w(y,t)dy + \lambda(1;\hat{q}_1(0),\hat{q}_2(t))\zeta(t)\right], \tag{94}$$

$t \in [0, \lim_{i\to\infty}(\tau_i))$. If $\zeta(0) = 0$, then $z[t], w[t], \zeta(t)$ are identically zero on $t \in [0, \lim_{i\to\infty}(\tau_i))$ according to (91)–(93). Next, we discuss the case of $\zeta(0) \neq 0$. Considering $z[t] \equiv 0$ on $t \in [0, \lim_{i\to\infty}(\tau_i))$, the dynamics for $w[t], \zeta(t)$ given as (91)–(94), and the definition of triggering times (38), (40), (41), we have that $\lim_{i\to\infty}(\tau_i) > \frac{1}{q_2}$. The equation $w(0,t) = \frac{c}{p}\zeta(t)$ (93) holding for $t \in [0, \lim_{i\to\infty}(\tau_i))$ requires the initial condition of w to be $w(x,0) = \frac{c}{p}e^{[(a-q_1bc)+b(q_2+q_1p)\frac{c}{p}]\frac{1}{q_2}x}\zeta(0)$, for ensuring that (93) holds on $t \in [0, \frac{1}{q_2}]$, and $w(1,t)$ to be

$$w(1,t) = \frac{c}{p}e^{[(a-q_1bc)+b(q_2+q_1p)\frac{c}{p}]\frac{1}{q_2}}\zeta(t), \quad t \in [0, \lim_{i\to\infty}(\tau_i)), \tag{95}$$

for ensuring that (93) holds on $t \in [\frac{1}{q_2}, \lim_{i\to\infty}(\tau_i))$.

Comparing (95), where $w(1,t)$ is a continuous function by virtue of (91), with (94) which includes possible discontinuities in $\hat{q}_2$, the necessary condition for the Equation (93) to hold on $t \in [0, \lim_{i\to\infty}(\tau_i))$ is that $w(1,t)$ is a continuous function. In other words, there is no discontinuity in Case 2. Considering the state of the w-PDE in (92) propagates from $x=1$ to $x=0$ with the propagation speed q_2, by representing the function $w(y,t)$ as the future value of $\zeta(t)$, using the expression for $\zeta(t)$ given by (91), then the relation (94) is written as

$$\begin{aligned} w(1,t) = \frac{\hat{q}_2(t)}{q_2}\Bigg[\int_0^1 \phi(1,y;\hat{q}_1(0),\hat{q}_2(t))\frac{c}{p}e^{[(a-q_1bc)+b(q_2+q_1p)\frac{c}{p}]\frac{1}{q_2}y}dy \\ + \lambda(1;\hat{q}_1(0),\hat{q}_2(t))\Bigg]\zeta(t), \ t \geq 0. \end{aligned} \tag{96}$$

Comparing (95) and (96), applying (15), (16), the necessary condition of $w(0,t) = \frac{c}{p}\zeta(t)$ (93) always holds on $t \in [\frac{1}{q_2}, \lim_{i\to\infty}(\tau_i))$ (when $\zeta(0) \neq 0$), is

$$\begin{aligned} \frac{1}{q_2}\Bigg[\int_0^1 \frac{c\kappa b}{p}e^{\frac{1}{\hat{q}_2(t)}(a-\hat{q}_1(0)bc)(1-y)}e^{[(a-q_1bc)+b(q_2+q_1p)\frac{c}{p}]\frac{1}{q_2}y}dy \\ + \frac{\hat{q}_2(t)\kappa}{\hat{q}_1(0)p+\hat{q}_2(t)}e^{\frac{1}{\hat{q}_2(t)}(a-\hat{q}_1(0)bc)}\Bigg] \equiv \frac{c}{p}e^{[(a-q_1bc)+b(q_2+q_1p)\frac{c}{p}]\frac{1}{q_2}}, \end{aligned} \tag{97}$$

on $t \in [0, \lim_{i\to\infty}(\tau_i))$. The right-hand side of (97) is constant while the left-hand side of (97) includes $\hat{q}_2(t)$, whose potential values are $q_2, \hat{q}_2(0)$ because of Lemma 3. If the left-hand side of (97) is varying with $\hat{q}_2(t)$, then (97) does not hold. If the left-hand side of (97) is kept constant with $\hat{q}_2(t) = q_2$ and $\hat{q}_2(t) = \hat{q}_2(0)$ (such as $\hat{q}_2(0) = q_2$), since, as we mentioned above, there is no discontinuity in Case 2, (97) holds only when the design parameter κ is equal to κ^*, where

$$\begin{aligned} \kappa^* = \frac{cq_2}{p}e^{[(a-q_1bc)+b(q_2+q_1p)\frac{c}{p}]\frac{1}{q_2}} \\ \div\left(\int_0^1 \frac{bc}{p}e^{\frac{1}{q_2}(a-\hat{q}_1(0)bc)(1-y)}e^{[(a-q_1bc)+b(q_2+q_1p)\frac{c}{p}]\frac{1}{q_2}y}dy + \frac{q_2e^{\frac{1}{q_2}(a-\hat{q}_1(0)bc)}}{\hat{q}_1(0)p+q_2}\right), \end{aligned} \tag{98}$$

where the symbol $\div$ denotes division. The constant κ^* is positive because $b>0$, $c>0$, $p>0$, $q_1>0$, $q_2>0$, $\hat{q}_1(0)>0$. The positivity of $\kappa=\kappa^*$ contradicts $\kappa<0$. Therefore, Case 2 would happen only when $\zeta(0)=0$, where $\Omega(t)\equiv 0$ on $t\in[0,\lim_{i\to\infty}(\tau_i))$. Therefore, $\tau_j=jT, j\in Z_+$.

Case 3. If neither of $\hat{q}_1(t),\hat{q}_2(t)$ reach q_1,q_2, it follows that $z[t], w[t], \zeta(t)$ are identically zero on $t\in[0,\lim_{i\to\infty}(\tau_i))$, that is, $\Omega(t)\equiv 0$ on $t\in[0,\lim_{i\to\infty}(\tau_i))$, according to Claim 4 and (4). Therefore, $\tau_j=jT, j\in Z_+$ according to (38) and (41).

By virtue of the results in the above discussions, we have that $\lim_{i\to\infty}(\tau_i)=+\infty$. The well-posedness of the closed-loop system is then obtained by recalling Proposition 1 and $\lim_{i\to\infty}(\tau_i)=+\infty$. This completes the proof of portion (1) of the theorem. The fact that $\lim_{i\to\infty}(\tau_i)=+\infty$ allows that the solution is defined on $\mathbb{R}_+$.

(2) Now we prove the second of the three portions of the theorem. Recalling the results in the discussions in Cases 1–3 in the proof of portion (1), and $\lim_{i\to\infty}(\tau_i)=+\infty$, we conclude that $\Omega(t)$ reaches zero not latter than $\frac{1}{q_1}$, that is, $\Omega(t)\equiv 0$ on $t\in[\frac{1}{q_1},\infty)$, when the finite time convergence of the parameter estimates to the true values is not achieved. Thus, portion (2) of the theorem is obtained.

(3) Finally, we prove the last of the three portions of the theorem, that is, establishing the exponential regulation result when estimates $(\hat{q}_1(t),\hat{q}_2(t))$ reach the true values (q_1,q_2) in finite time τ_ε, that is, when

$$\hat{q}_1(t)=q_1,\quad \hat{q}_2(t)=q_2,\quad t\geq\tau_\varepsilon. \tag{99}$$

Define a Lyapunov function

$$V(t)=\frac{1}{2}r_a\int_0^1 e^{\delta x}\beta(x,t)^2dx+\frac{1}{2}r_b\int_0^1 e^{-\delta x}\alpha(x,t)^2dx+\frac{1}{2}\zeta(t)^2,\quad t\geq 0, \tag{100}$$

where the positive constants r_a, r_b, δ are constrained through the inequalities (10), (9), (28). Denoting

$$\Omega_1(t)=\|\alpha[t]\|^2+\|\beta[t]\|^2+\zeta(t)^2,$$

we obtain

$$\xi_1\Omega_1(t)\leq V(t)\leq\xi_2\Omega_1(t),\quad t\geq 0, \tag{101}$$

where the positive constants ξ_1,ξ_2 are shown in (31) and (32).

Define the errors between the gains in the nominal control law (23) and those in the certainty-equivalence controller (36), caused by the parameter estimate errors, as

$$\tilde{q}_1(t)=q_1-\hat{q}_1(t), \tag{102}$$

$$\tilde{R}_1(y,t)=q_2\phi(1,y;q_1,q_2)-\hat{q}_2(t)\phi(1,y;\hat{q}_1(t),\hat{q}_2(t)), \tag{103}$$

$$\tilde{R}_2(t)=q_2\lambda(1;q_1,q_2)-\hat{q}_2(t)\lambda(1;\hat{q}_1(t),\hat{q}_2(t)), \tag{104}$$

where $\phi(1,y;\hat{q}_1(t),\hat{q}_2(t))$, $\lambda(1;\hat{q}_1(t),\hat{q}_2(t))$ are the results of replacing q_1,q_2 with $\hat{q}_1(t),\hat{q}_2(t)$ in $\phi(1,y;q_1,q_2)$ and $\lambda(1;q_1,q_2)$. Because of (99), $\tilde{q}_1(t)$, $\tilde{R}_1(\cdot,t)$, $\tilde{R}_2(t)$ are zero for $t\geq\tau_\varepsilon$.

Applying the adaptive control law (36), recalling (23), the boundary condition (21) in the target system (17)–(21) becomes

$$\beta(1,t)=\frac{1}{q_2}[\tilde{q}_1(t)z(1,t)-\int_0^1\tilde{R}_1(y,t)w(y,t)dy-\tilde{R}_2(t)\zeta(t)]. \tag{105}$$

Applying the Cauchy–Schwarz inequality into the backstepping transformation (13), (14) and its inverse

$$z(x,t)=\alpha(x,t), \tag{106}$$

$$w(x,t)=\beta(x,t)-\int_0^x\frac{\kappa b}{q_2}e^{\frac{m}{q_2}(x-y)}\beta(y,t)dy-\frac{\kappa}{(q_2+q_1p)}e^{\frac{m}{q_2}x}\zeta(t), \tag{107}$$

we have that $\Omega_1(t)$ is bounded by

$$\xi_3\Omega(t) \leq \Omega_1(t) \leq \xi_4\Omega(t), \quad t \geq 0, \tag{108}$$

where the positive constants ξ_3, ξ_4 are defined by (33)–(35).

Taking the derivative of (100) along (17)–(21), (105), applying Young's inequality and the Cauchy–Schwarz inequality, we get

$$\begin{aligned}\dot{V}(t) \leq & -\left(\frac{1}{2}m - q_1 r_b c_0^2\right)\zeta(t)^2 - \left(\frac{q_2 r_a}{2} - q_1 r_b p^2 - \frac{(q_1 p + q_2)^2 b^2}{2m}\right)\beta(0,t)^2 \\ & - \frac{1}{2}r_a\delta q_2\int_0^1 e^{\delta x}\beta(x,t)^2dx - \frac{1}{2}r_b\delta q_1\int_0^1 e^{-\delta x}\alpha(x,t)^2dx \\ & - \frac{1}{2}q_1 r_b e^{-\delta}\alpha(1,t)^2 + \frac{1}{2}q_2 r_a e^{\delta}\beta(1,t)^2,\end{aligned} \tag{109}$$

for $t \geq 0$. Recalling (9) and (10), then (109) becomes

$$\dot{V}(t) \leq -\lambda_1 V(t) + \frac{1}{2}q_2 r_a e^{\delta}\beta(1,t)^2, \tag{110}$$

for $t \geq 0$, where the positive constant λ_1 is shown in (27). Multiplying both sides of (110) by $e^{\lambda_1 t}$, yields

$$\frac{d(V(t)e^{\lambda_1 t})}{dt} \leq \frac{1}{2}e^{\lambda_1 t}q_2 r_a e^{\delta}\beta(1,t)^2, \quad t \geq 0, \tag{111}$$

and then, integrating from τ_ε to t, we obtain

$$V(t)e^{\lambda_1 t} - V(\tau_\varepsilon)e^{\lambda_1\tau_\varepsilon} \leq \int_{\tau_\varepsilon}^t \frac{1}{2}e^{\lambda_1\varsigma}q_2 r_a e^{\delta}\beta(1,\varsigma)^2 d\varsigma, \quad t \geq \tau_\varepsilon. \tag{112}$$

Recalling (105) and the fact that $\tilde{q}_1(t)$, $\tilde{R}_1(t)$, $\tilde{R}_2(t)$ are identically zero for $t \geq \tau_\varepsilon$, we get $\beta(1,t) \equiv 0$ for $t \geq \tau_\varepsilon$ according to (105). Therefore, the term $\int_{\tau_\varepsilon}^t \frac{1}{2}e^{\lambda_1\varsigma}q_2 r_a e^{\delta}\beta(1,\varsigma)^2 d\varsigma$ in (112) is zero. Multiplying both sides of (112) by $e^{-\lambda_1 t}$, yields

$$V(t) \leq V(\tau_\varepsilon)e^{-\lambda_1(t-\tau_\varepsilon)}, \quad t \geq \tau_\varepsilon. \tag{113}$$

Recalling (101), we get

$$\Omega_1(t) \leq \frac{\xi_2}{\xi_1}\Omega_1(\tau_\varepsilon)e^{-\lambda_1(t-\tau_\varepsilon)}, \quad t \geq \tau_\varepsilon.$$

Recalling (108), we further have that

$$\Omega(t) \leq \Upsilon_\theta\Omega(\tau_\varepsilon)e^{-\lambda_1(t-\tau_\varepsilon)}, \quad t \geq \tau_\varepsilon, \tag{114}$$

where the overshoot coefficient Υ_θ is shown in (30).

If $\tau_\varepsilon = 0$, we obtain directly from (114) that $\Omega(t) \leq \Upsilon_\theta\Omega(0)e^{-\lambda_1 t}, t \geq 0$. Next, we conduct analysis for $t \in [0, \tau_\varepsilon]$ when $\tau_\varepsilon \neq 0$. Recalling (13), (105), (109), (110), we obtain

$$\dot{V}(t) \leq -\lambda_1 V(t) - \left(\frac{1}{2}q_1 r_b e^{-\delta} - \frac{1}{q_2}r_a e^{\delta}(\overline{q}_1 - \underline{q}_1)^2\right)\alpha(1,t)^2 + \frac{9r_a e^{\delta}}{2q_2}\left[\int_0^1 \tilde{R}_1(y,t)^2 w(y,t)^2 dy + \tilde{R}_2(t)^2\zeta(t)^2\right], \tag{115}$$

for $t \in [0, \tau_\varepsilon]$. Recalling (28), which makes the coefficient in the parentheses in front of $\alpha(1,t)^2$ positive, and recalling $\tilde{R}_1(y,t)$, $\tilde{R}_2(t)$, defined by (103)–(104), where $\hat{q}_1(t)$ is equal to either $\hat{q}_1(0)$ or q_1 and $\hat{q}_2(t)$ is equal to either $\hat{q}_2(0)$ or q_2 in $t \in [0, \tau_\varepsilon]$ according to Lemma 3, as well as applying (107) and the Cauchy–Schwarz inequality, we obtain from (115) that

$$\dot{V}(t) \leq -\lambda_1 V(t) + Q(\hat{q}_1(0), \hat{q}_2(0), q_1, q_2)V(t), \quad t \in [0, \tau_\varepsilon], \tag{116}$$

where the positive constant $Q(\hat{q}_1(0), \hat{q}_2(0), q_1, q_2)$, obtained by bounding the last line of (115), is a family of constants parameterized by positive constants q_1, q_2, $\hat{q}_1(0)$, $\hat{q}_2(0)$.

If $\lambda_1 < Q(\hat{q}_1(0), \hat{q}_2(0), q_1, q_2)$, defining a positive constant

$$\lambda_2(\hat{q}_1(0), \hat{q}_2(0), q_1, q_2) = Q(\hat{q}_1(0), \hat{q}_2(0), q_1, q_2) - \lambda_1, \tag{117}$$

multiplying both sides of (116) by $e^{-\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)t}$, we obtain

$$\dot{V}(t)e^{-\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)t} - \lambda_2(\hat{q}_1(0), \hat{q}_2(0), q_1, q_2)V(t)e^{-\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)t} \leq 0, \tag{118}$$

for $t \in [0, \tau_\varepsilon]$, that is,

$$\frac{d(V(t)e^{-\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)t})}{dt} \leq 0, \tag{119}$$

for $t \in [0, \tau_\varepsilon]$. Then, integrating from 0 to t, yields

$$V(t) \leq V(0)e^{\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)t}, \quad t \in [0, \tau_\varepsilon]. \tag{120}$$

Recalling (101), (108), we get

$$\Omega(t) \leq \Upsilon_\theta \Omega(0)e^{\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)t}, \quad t \in [0, \tau_\varepsilon]. \tag{121}$$

If $\lambda_1 \geq Q(q_1(0), q_2(0), q_1, q_2)$, by defining a positive constant

$$\lambda_3(\hat{q}_1(0), \hat{q}_2(0), q_1, q_2) = \lambda_1 - Q(\hat{q}_1(0), \hat{q}_2(0), q_1, q_2), \tag{122}$$

we obtain from (116) that

$$\Omega(t) \leq \Upsilon_\theta \Omega(0)e^{-\lambda_3(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)t}, \quad t \in [0, \tau_\varepsilon]. \tag{123}$$

Comparing (121) and (123), we obtain

$$\Omega(\tau_\varepsilon) \leq \Upsilon_\theta e^{\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)\tau_\varepsilon}\Omega(0), \quad t \in [0, \tau_\varepsilon]. \tag{124}$$

Let us now recap that Assumption 2, that is, (8), is only used to ensure the existence of a positive δ satisfying (28), which enables going from (115) to (116), with the purpose to arrive at (124). In plain words, Assumption 2 is only used in the Lyapunov analysis when the estimate $\hat{q}_1(t)$ has not reached the true value q_1, in order to ensure (124).

Recalling (114), we conclude that

$$\Omega(t) \leq \Upsilon_\theta^2 e^{\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)\tau_\varepsilon} e^{\lambda_1 \tau_\varepsilon}\Omega(0)e^{-\lambda_1 t}, \quad t \geq 0. \tag{125}$$

Claim 5. If $\hat{\theta}(t)$ reaches θ at τ_ε, then $\tau_\varepsilon \leq \max\{\frac{1}{q_2} + T, 2T\}$.

Proof. The proof is shown in Appendix F. ■

Applying Claim 5, (125) is written as

$$\Omega(t) \leq \Upsilon_\theta^2 e^{\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)\max\{\frac{1}{q_2}+T,2T\}} e^{\lambda_1 \max\{\frac{1}{q_2}+T,2T\}}\Omega(0)e^{-\lambda_1 t}, \tag{126}$$

for $t \geq 0$. Denoting

$$M_{\theta,\hat{\theta}(0)} = \Upsilon_\theta^2 e^{\lambda_2(\hat{q}_1(0),\hat{q}_2(0),q_1,q_2)\max\{\frac{1}{q_2}+T,2T\}} e^{\lambda_1 \max\{\frac{1}{q_2}+T,2T\}},$$

we obtain (87). This completes the proof of portion (3) of the theorem.

With the proof of Theorem 1 completed, we thank the reader for sticking with us for the nearly six-page ride and commend the reader's stamina. ■

6 | SIMULATION

The simulation is conducted for the plant (1)–(5) with the model parameters taken as

$$a = 13,\ b = 1,\ c = 2,\ p = 0.5, \tag{127}$$

$$\overline{c} = 1,\ q_1 = 4,\ q_2 = 6, \tag{128}$$

where q_1, q_2 are treated as unknown, with the known bounds

$$\overline{q}_1 = 6,\ \underline{q}_1 = 2,\ \overline{q}_2 = 7,\ \underline{q}_2 = 3. \tag{129}$$

The initial values for the test are chosen as

$$z(x,0) = \cos\left(\pi x + \frac{\pi}{4}\right) + x^3, \tag{130}$$

$$w(x,0) = \sin\left(1.5\pi x + \frac{\pi}{3}\right) + x^2, \tag{131}$$

$$\zeta(0) = \frac{1}{c} z(0,0) + \frac{p}{c} w(0,0). \tag{132}$$

The finite difference method is adopted to conduct the simulation with the time and space steps of 0.0001 and 0.01, respectively.

For the regulation-triggered batch least-squares identifier defined by (37), (38), (40), (41), (47), (61), we choose $n = 1, 2, \dots, 7$ and

$$\overline{a} = 0.8,\ \tilde{N} = 1,\ T = 8, \tag{133}$$

and take the initial values of the estimates as $\hat{q}_1(0) = \underline{q}_1$ $\hat{q}_2(0) = \underline{q}_2$, namely, we start the parameter estimates from their lower bounds. Using the given bounds in (129) to determine the gain κ in the controller (36) by (12), a control gain satisfying $\kappa < -267$ is needed. For $\kappa = -320$, recalling the model parameters in (127) and (128), we obtain $r_b < 0.079$, $r_a > 0.06$ according to (9), (10), which indicates that $\overline{q}_1 - \underline{q}_1$ needs to be smaller than 4.01 according to Assumption 2 (satisfied by the known bounds of q_1 given in (129)). The source of the high gain κ is the first term of (12) which is used in Claim 3 to exclude some rare and extreme situations ($w(x,t) = M$ and $z(x,t) = -\overline{\lambda} M$ for $x \in [0,1]$, $t \in [\mu_{i+1}, \tau_{i+1}]$ where M, $\overline{\lambda}$ are nonzero constants) affecting the exact parameter estimation. In the simulation, we find the high gain is actually not needed (the aforementioned extreme situations do not happen) and $\kappa = -6$ derived from the second term in (12) is perfectly sufficient to achieve a satisfactory result. According to the initial values of the estimates $\hat{q}_1(0)$ $\hat{q}_2(0)$, we get a large initial value $\Upsilon_{\hat{\theta}(0)}$ by (30)–(35), (9), (10), which is a conservative value obtained by the stability analysis in Section 5. In this simulation, guided by reason rather than by a highly conservative estimate, we adopt a smaller initial value as $\Upsilon_{\hat{\theta}(0)} = 5.5$, which prevents the activation of the identifer from being extremely late (particularly relative to T).

From Figure 2, we observe that the estimates $\hat{q}_1, \hat{q}_2$ reach the exact values of the unknown parameters $q_1 = 4, q_2 = 6$ at $t = 0.13$ s, in just one trigger. In Figure 3, the nominal control input applied at the boundary $x = 1$ goes through the PDE domain and reaches the boundary $x = 0$, starting to regulate the ODE state $\zeta(t)$ at $t = \frac{1}{q_2} \approx 0.17$s. As shown in Figure 2, the estimates reach the true values and update the certainty-equivalence controller at $t = 0.13$s. Then it takes $1/q_2 \approx 0.17$s

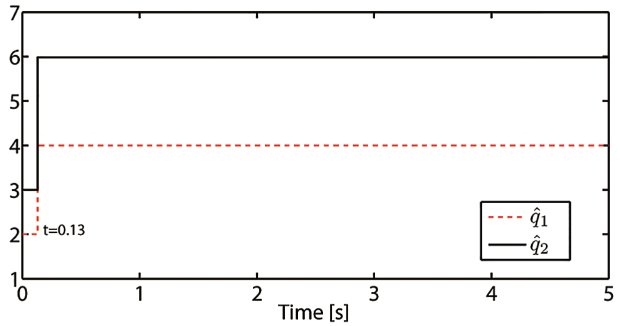

FIGURE 2 Parameter estimates $\hat{q}_1(t), \hat{q}_2(t)$

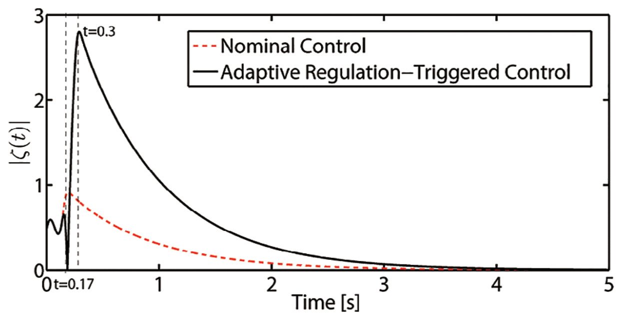

FIGURE 3 The evolution of $|\zeta(t)|$ under the nominal control (23) and the proposed adaptive regulation-triggered control (36)

for the updated control signal to travel to the ODE, that is, the updated control signal starts to properly regulate the ODE state $\zeta(t)$, as intended by the nominal controller, at $t = 0.3$s. For the remaining time, as shown in Figure 3, the performance of the proposed adaptive controller coincides with the nominal feedback, and $|\zeta(t)|$ converges to zero. Similar results are observed in Figure 4 which shows the evolution of $\Omega(t)^{\frac{1}{2}}$ defined by (24), under the nominal control and the proposed adaptive regulation-triggered control. Figures 5 and 6 show the PDE states $z(x,t), w(x,t)$ are regulated to zero under the proposed adaptive regulation-triggered controller. The adaptive regulation-triggered control law and the nominal control law are shown in Figure 7.

At the end of this section, and this paper, let us reiterate that, as announced at the beginning of this paper, that the BaLSI identifier has ensured the perfect identification of the unknown parameters in finite time and enabled the

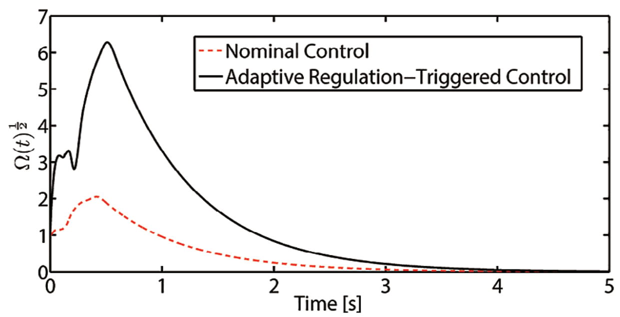

FIGURE 4 The evolution of $\Omega(t)^{\frac{1}{2}}$ under the nominal control (23) and the proposed adaptive regulation-triggered control (36)

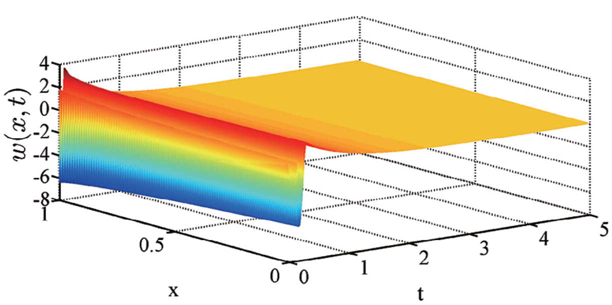

FIGURE 5 The evolution of $w(x,t)$ under the proposed adaptive regulation-triggered control (36)

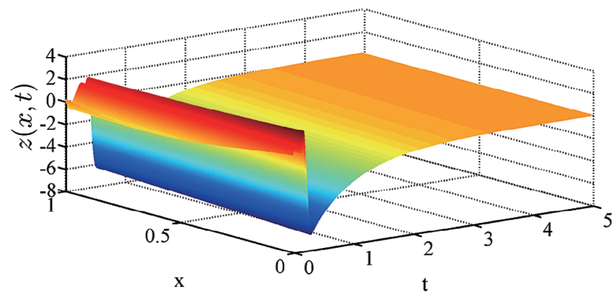

FIGURE 6 The evolution of $z(x, t)$ under the proposed adaptive regulation-triggered control (36)

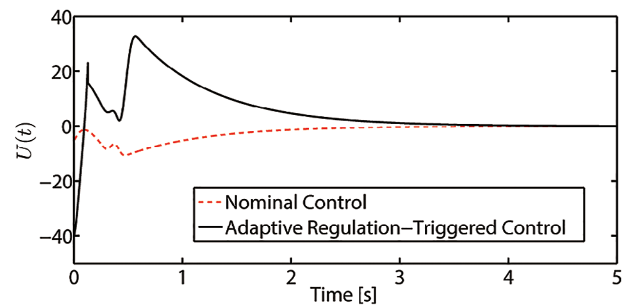

FIGURE 7 The control signals of the nominal control (23) and the proposed adaptive regulation-triggered control (36)

regulation-triggered adaptive backstepping controller to achieve exponential regulation, with a decay rate matching the rate corresponding to the case of known parameters.

7 | CONCLUSIONS

In this paper, we have proposed an adaptive boundary control scheme for a heterodirectional transport PDE-ODE system where both transport speeds are unknown. It is a certainty equivalence-based adaptive boundary control scheme using a batch least-squares identifier updated at a sequence of times, which are determined by an event trigger designed based on the progress of the regulation of the states. We have proved that the proposed triggering-based adaptive control guarantees: (1) the absence of a Zeno phenomenon; (2) parameter estimates are convergent to the true values in finite time (from most initial conditions); (3) exponential regulation of the plant states. The effectiveness of the proposed design is verified by a numerical example.

In future work, the state-feedback control design will be extended to the output-feedback type to meet the requirements of more engineering applications. While the present work considered model-based adaptive control of a string-ODE cascade, one could also bring to bear, in certain applications, the extremum seeking control algorithms in the presence of wave PDE dynamics, introduced in Reference 53. One application would be deep-sea cable-actuated source seeking, with a sensor, deprived of position awareness due to the undersea environment, hung on a cable, moved through the cable from the sea surface using a surface vessel, and tasked with being located as close as possible to a signal source. The algorithm in Reference 53 is applicable to such a source-seeking scenario and would, in addition to finding the signal source, stabilize the motion of the cable. The deeper the signal source, that is, the longer the cable, the easier the problem would become from the perspective of the surface vessel (the high natural frequency of the long cable would not necessitate rapid motion of the vessel), but the lengthier memory would be required in the PDE-compensating extremum seeking algorithm in Reference 53. Additionally, oil drilling penetration maximization, through the drill-string PDE dynamics, could be pursued in a model-free adaptive fashion as in Reference 54.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable since the article describes the theoretical research.

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APPENDIX A. THE PROOF OF PROPOSITION 1

Inserting (36) into (5), the closed-loop system is

$$\dot{\zeta}(t) = (a - q_1bc)\zeta(t) + b(q_2 + q_1p)w(0,t), \tag{A1}$$

$$z_t(x,t) = -q_1z_x(x,t), \tag{A2}$$

$$w_t(x,t) = q_2w_x(x,t), \tag{A3}$$

$$z(0,t) = c\zeta(t) - pw(0,t), \tag{A4}$$

$$w(1,t) = \frac{1}{q_2}\hat{q}_2(\tau_i)\int_0^1 \phi(1,y;\hat{\theta}(\tau_i))w(y,t)dy + \frac{1}{q_2}\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i))\zeta(t) + \frac{q_1 - \hat{q}_1(\tau_i)}{q_2}z(1,t), \tag{A5}$$

for $t \in [\tau_i, \tau_{i+1}), x \in [0,1]$, $i \in Z_+$, where $\hat{\theta}(\tau_i) = (\hat{q}_1(\tau_i), \hat{q}_2(\tau_i))^T$ is constant and defined by (37), (38), (40), (41), (47), (61). With the purpose of decoupling the ODE and the PDEs, we introduce two transformations. The first one is the following Volterra transformation:

$$\overline{z}(x,t) = z(x,t) - \int_0^x \overline{\phi}(x-y)z(y,t)dy - \overline{\varphi}(x)\zeta(t), \tag{A6}$$

where the functions $\overline{\varphi}$ and $\overline{\phi}$ satisfy

$$q_1\overline{\varphi}'(x) + \left((a - q_1bc) + \frac{b}{p}(q_2 + q_1p)c\right)\overline{\varphi}(x) = 0, \tag{A7}$$

$$\overline{\varphi}(0) = c, \tag{A8}$$

$$\overline{\phi}(x) = \frac{1}{q_1p}\overline{\varphi}(x)b(q_2 + q_1p). \tag{A9}$$

Through the transformation (A6), (A16), the system (A1)–(A5) is converted to

$$\dot{\zeta}(t) = (a - q_1bc)\zeta(t) + b(q_2 + q_1p)w(0,t), \tag{A10}$$

$$\overline{z}_t(x,t) = -q_1\overline{z}_x(x,t), \tag{A11}$$

$$w_t(x,t) = q_2w_x(x,t), \tag{A12}$$

$$\overline{z}(0,t) = -pw(0,t), \tag{A13}$$

$$\begin{aligned} w(1,t) = {} & \frac{q_1 - \hat{q}_1(\tau_i)}{q_2}\overline{z}(1,t) + \frac{1}{q_2}\hat{q}_2(\tau_i)\int_0^1 \phi(1,y;\hat{\theta}(\tau_i))w(y,t)dy - \frac{q_1 - \hat{q}_1(\tau_i)}{q_2}\int_0^1 \overline{\psi}(1-y)\overline{z}(y,t)dy, \\ & + \left[\frac{1}{q_2}\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)) - \frac{q_1 - \hat{q}_1(\tau_i)}{q_2}\overline{\gamma}(1)\right]\zeta(t), \end{aligned} \tag{A14}$$

for $t \in [\tau_i, \tau_{i+1}), x \in [0,1]$. The conditions (A7)–(A9) of the functions $\overline{\varphi}$ and $\overline{\phi}$ in the transformation (A6), (A16) are obtained through matching (A1)–(A5) and (A10)–(A14), as follows. Inserting (A6) into (A11), using (A1), (A2), (A4), we obtain

$$\begin{aligned} \overline{z}_t(x,t) + q_1\overline{z}_x(x,t) = {} & z_t(x,t) + q_1\int_0^x \overline{\phi}(x-y)z_x(y,t)dy - \overline{\varphi}(x)(a - q_1bc)\zeta(t) \\ & - \overline{\varphi}(x)b(q_2 + q_1p)w(0,t) + q_1z_x(x,t) - q_1\overline{\phi}(0)z(x,t) \end{aligned}$$

$$
\begin{aligned}
&- \int_0^x q_1\overline{\phi}'(x-y)z(y,t)dy - q_1\overline{\varphi}'(x)\zeta(t) \\
&= q_1\overline{\phi}(0)z(x,t) - q_1\overline{\phi}(x)z(0,t) - q_1\overline{\phi}(0)z(x,t) - q_1\overline{\varphi}'(x)\zeta(t) \\
&- \overline{\varphi}(x)(a-q_1bc)\zeta(t) - \frac{1}{p}\overline{\varphi}(x)b(q_2+q_1p)c\zeta(t) + \frac{1}{p}\overline{\varphi}(x)b(q_2+q_1p)z(0,t) \\
&= \left[\frac{1}{p}\overline{\varphi}(x)b(q_2+q_1p) - q_1\overline{\phi}(x)\right] z(0,t) - [\overline{\varphi}(x)(a-q_1bc) + \frac{1}{p}\overline{\varphi}(x)b(q_2+q_1p)c + q_1\overline{\varphi}'(x)]\zeta(t) = 0.
\end{aligned} \tag{A15}
$$

For (A15) to hold, we obtain the conditions (A7), (A9). Matching (A4) and (A13), we get the condition (A8). Because $\overline{\phi}$ is a continuous function, we have that the inverse transformation

$$
z(x,t) = \overline{z}(x,t) - \int_0^x \overline{\psi}(x-y)\overline{z}(y,t)dy - \overline{\gamma}(x)\zeta(t), \tag{A16}
$$

exists (see, e.g., chapter 9.9 in Reference 55), where the well-posedness of $\overline{\psi}$, $\overline{\gamma}$ is ensured by the well-posedness of (A7)–(A9).

Applying the second transformation

$$
\chi(t) = \zeta(t) - \int_0^1 K_{1i}(x)\overline{z}(x,t)dx - \int_0^1 K_{2i}(x)w(x,t)dx, \tag{A17}
$$

for $t \in [\tau_i, \tau_{i+1})$, $i \in Z_+$, where the functions K_{1i}, K_{2i} satisfy the following well-posed first-order ODEs:

$$
[(a-q_1bc) + K_{2i}(1)((q_1-\hat{q}_1(\tau_i))\overline{\gamma}(1) - \hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)))]K_{1i}(x) - q_1K_{1i}{}'(x) = -K_{2i}(1)(q_1-\hat{q}_1(\tau_i))\overline{\psi}(1-x), \tag{A18}
$$

$$
[(a-q_1bc) + K_{2i}(1)((q_1-\hat{q}_1(\tau_i))\overline{\gamma}(1) - \hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)))]K_{2i}(x) + q_2K_{2i}{}'(x) = K_{2i}(1)\hat{q}_2(\tau_i)\phi(1,x;\hat{\theta}(\tau_i)), \tag{A19}
$$

$$
K_{1i}(1)q_1 = K_{2i}(1)(q_1-\hat{q}_1(\tau_i)), \tag{A20}
$$

$$
K_{2i}(0)q_2 = -b(q_2+q_1p) - pK_{1i}(0)q_1, \tag{A21}
$$

the system (A10)–(A14) is transformed to

$$
\dot{\chi}(t) = A_i\chi(t), \tag{A22}
$$

$$
\overline{z}_t(x,t) = -q_1\overline{z}_x(x,t), \tag{A23}
$$

$$
w_t(x,t) = q_2w_x(x,t), \tag{A24}
$$

$$
\overline{z}(0,t) = -pw(0,t), \tag{A25}
$$

$$
w(1,t) = \frac{q_1-\hat{q}_1(\tau_i)}{q_2}\overline{z}(1,t) + \int_0^1 D_{1i}(x)w(x,t)dx + \int_0^1 D_{2i}(x)\overline{z}(x,t)dx + D_{3i}\chi(t), \tag{A26}
$$

for $t \in [\tau_i, \tau_{i+1}), x \in [0,1]$, where

$$
\begin{aligned}
A_i &= a - q_1bc + K_{2i}(1)(q_1-\hat{q}_1(\tau_i))\overline{\gamma}(1) - K_{2i}(1)\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)), \\
D_{1i}(x) &= \frac{1}{q_2}\hat{q}_2(\tau_i)\phi(1,x;\hat{\theta}(\tau_i)) + \left(\frac{1}{q_2}\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)) - \frac{q_1-\hat{q}_1(\tau_i)}{q_2}\overline{\gamma}(1)\right)K_{1i}(x), \\
D_{2i}(x) &= -\frac{q_1-\hat{q}_1(\tau_i)}{q_2}\overline{\psi}(1-x) + \left(\frac{1}{q_2}\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)) - \frac{q_1-\hat{q}_1(\tau_i)}{q_2}\overline{\gamma}(1)\right)K_{2i}(x), \\
D_{3i} &= \frac{1}{q_2}\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)) - \frac{q_1-\hat{q}_1(\tau_i)}{q_2}\overline{\gamma}(1).
\end{aligned}
$$

The conditions (A18)–(A21) of $K_{1i}(x)$, $K_{2i}(x)$ are defined by matching (A10)–(A14) and (A22)–(A26), as follows. Inserting (A17) into (A22), using (A10)–(A14), we obtain that

$$\begin{aligned}
&\dot{\chi}(t) - A_i\chi(t) \\
&= \dot{\chi}(t) - (a - q_1bc)\chi(t) - (K_{2i}(1)(q_1 - \hat{q}_1(\tau_i))\overline{\gamma}(1) - K_{2i}(1)\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)))\chi(t) \\
&= \dot{\zeta}(t) - \int_0^1 K_{1i}(x)\overline{z}_t(x,t)dx - \int_0^1 K_{2i}(x)w_t(x,t)dx \\
&\quad - (a - q_1bc)\zeta(t) + (a - q_1bc)\int_0^1 K_{1i}(x)\overline{z}(x,t)dx \\
&\quad + (a - q_1bc)\int_0^1 K_{2i}(x)w(x,t)dx - (K_{2i}(1)(q_1 - \hat{q}_1(\tau_i))\overline{\gamma}(1) - K_{2i}(1)\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)))\chi(t) \\
&= b(q_2 + q_1p)w(0,t) + q_1\int_0^1 K_{1i}(x)\overline{z}_x(x,t)dx \\
&\quad - q_2\int_0^1 K_{2i}(x)w_x(x,t)dx + (a - q_1bc)\int_0^1 K_{1i}(x)\overline{z}(x,t)dx \\
&\quad + (a - q_1bc)\int_0^1 K_{2i}(x)w(x,t)dx - (K_{2i}(1)(q_1 - \hat{q}_1(\tau_i))\overline{\gamma}(1) - K_{2i}(1)\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)))\chi(t) \\
&= b(q_2 + q_1p)w(0,t) + K_{1i}(1)q_1\overline{z}(1,t) - K_{1i}(0)q_1\overline{z}(0,t) - \int_0^1 q_1K_{1i}{}'(x)\overline{z}(x,t)dx \\
&\quad - K_{2i}(1)q_2w(1,t) + K_{2i}(0)q_2w(0,t) + \int_0^1 q_2K_{2i}{}'(x)w(x,t)dx \\
&\quad + (a - q_1bc)\int_0^1 K_{1i}(x)\overline{z}(x,t)dx + (a - q_1bc)\int_0^1 K_{2i}(x)w(x,t)dx \\
&\quad - (K_{2i}(1)(q_1 - \hat{q}_1(\tau_i))\overline{\gamma}(1) - K_{2i}(1)\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)))\chi(t) \\
&= -K_{2i}(1)q_2[\frac{1}{q_2}\hat{q}_2(\tau_i)\int_0^1 \phi(1,y;\hat{\theta}(\tau_i))w(y,t)dy + \left(\frac{1}{q_2}\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)) - \frac{q_1 - \hat{q}_1(\tau_i)}{q_2}\overline{\gamma}(1)\right)\zeta(t) \\
&\quad + \frac{q_1 - \hat{q}_1(\tau_i)}{q_2}\overline{z}(1,t) - \frac{q_1 - \hat{q}_1(\tau_i)}{q_2}\int_0^1 \overline{\psi}(1-y)\overline{z}(y,t)dy] + K_{1i}(1)q_1\overline{z}(1,t) \\
&\quad + (K_{2i}(0)q_2 + b(q_2 + q_1p) + pK_{1i}(0)q_1)w(0,t) \\
&\quad + \int_0^1 (q_2K_{2i}{}'(x) + (a - q_1bc)K_{2i}(x))w(x,t)dx + \int_0^1 ((a - q_1bc)K_{1i}(x) - q_1K_{1i}{}'(x))\overline{z}(x,t)dx \\
&\quad - (K_{2i}(1)(q_1 - \hat{q}_1(\tau_i))\overline{\gamma}(1) - K_{2i}(1)\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)))\chi(t) \\
&= (K_{1i}(1)q_1 - K_{2i}(1)(q_1 - \hat{q}_1(\tau_i)))\overline{z}(1,t) + (K_{2i}(0)q_2 + b(q_2 + q_1p) + pK_{1i}(0)q_1)w(0,t) \\
&\quad + \int_0^1 [q_2K_{2i}{}'(x) + (a - q_1bc)K_{2i}(x) - K_{2i}(1)\hat{q}_2(\tau_i)\phi(1,x;\hat{\theta}(\tau_i)) + (K_{2i}(1)(q_1 - \hat{q}_1(\tau_i))\overline{\gamma}(1) \\
&\quad - K_{2i}(1)\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)))K_{2i}(x)]w(x,t)dx + \int_0^1 [(a - q_1bc)K_{1i}(x) - q_1K_{1i}{}'(x) \\
&\quad + K_{2i}(1)(q_1 - \hat{q}_1(\tau_i))\overline{\psi}(1-x) + (K_{2i}(1)(q_1 - \hat{q}_1(\tau_i))\overline{\gamma}(1) \\
&\quad - K_{2i}(1)\hat{q}_2(\tau_i)\lambda(1;\hat{\theta}(\tau_i)))K_{1i}(x)]\overline{z}(x,t)dx = 0.
\end{aligned} \tag{A27}$$

For (A27) to hold, the conditions (A18)–(A21) are obtained.

The equation set (A23)–(A26), where $\chi(t)$ is a well-defined external signal generated by (A22), has an analogous structure with (5), (6) in Reference 56. According to the result in the part 1) in appendix of Reference 56, we have that the system (A22)–(A26) has a unique solution on $t \in [\tau_i, \tau_{i+1})$ for all $(w[\tau_i], \overline{z}[\tau_i])^T \in L^2((0,1);\mathbb{R}^2)$, $\chi(\tau_i) \in \mathbb{R}$. By virtue of the transformations (A17), (A16), we obtain that, for given $(w[\tau_i], z[\tau_i])^T \in L^2((0,1);\mathbb{R}^2)$, $\zeta(\tau_i) \in \mathbb{R}$, the system (A1)–(A5) has a unique solution for $t \in [\tau_i, \tau_{i+1})$. Recalling the definition of the weak solution in Definition 1, we obtain that for every $(z[\tau_i], w[\tau_i])^T \in L^2((0,1);\mathbb{R}^2)$, $\zeta(\tau_i) \in \mathbb{R}$ and $\hat{\theta}(\tau_i) \in \Theta$, there exists a unique (weak) solution $((z,w)^T, \zeta) \in C^0([\tau_i, \tau_{i+1}]; L^2(0,1);\mathbb{R}^2) \times C^0([\tau_i, \tau_{i+1}];\mathbb{R})$ to the system (1)–(5) with (36), (37), (38), (40), (41), (47), (61).

For every $(z_0, w_0)^T \in L^2((0,1);\mathbb{R}^2)$, $\zeta_0 \in \mathbb{R}$ and $\hat{\theta}_0 \in \Theta$, through iterative constructions between successive triggering times, the proposition is thus obtained.

APPENDIX B. PROOF OF CLAIM 1

If $\overline{\lambda} = 0$, it means that $Q_{n_2,2}(\mu_{i+1}, \tau_{i+1}) = 0$ by recalling (85) and (56), and then $\ell_2 = \frac{H_{n_2,2}(\mu_{i+1},\tau_{i+1})}{Q_{n_2,3}(\mu_{i+1},\tau_{i+1})}$ in (78). Together with (77), we get

$$S_i = \left\{ \left(\frac{H_{n_1,1}(\mu_{i+1}, \tau_{i+1})}{Q_{n_1,1}(\mu_{i+1}, \tau_{i+1})} - \frac{H_{n_2,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_2,3}(\mu_{i+1}, \tau_{i+1})} \frac{Q_{n_1,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_1,1}(\mu_{i+1}, \tau_{i+1})}, \frac{H_{n_2,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_2,3}(\mu_{i+1}, \tau_{i+1})} \right) \right\}, \tag{B1}$$

is a singleton: a contradiction. Similarly, if $\overline{\lambda}_1 = 0$, it means that $Q_{n_1,2}(\mu_{i+1}, \tau_{i+1}) = 0$ by recalling (84), (56), and then $\ell_1 = \frac{H_{n_1,1}(\mu_{i+1},\tau_{i+1})}{Q_{n_1,1}(\mu_{i+1},\tau_{i+1})}$ in (77). Together with (78), we get

$$S_i = \left\{ \left(\frac{H_{n_1,1}(\mu_{i+1}, \tau_{i+1})}{Q_{n_1,1}(\mu_{i+1}, \tau_{i+1})}, \frac{H_{n_2,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_2,3}(\mu_{i+1}, \tau_{i+1})} - \frac{H_{n_1,1}(\mu_{i+1}, \tau_{i+1})}{Q_{n_1,1}(\mu_{i+1}, \tau_{i+1})} \frac{Q_{n_2,2}(\mu_{i+1}, \tau_{i+1})}{Q_{n_2,3}(\mu_{i+1}, \tau_{i+1})} \right) \right\}, \tag{B2}$$

is a singleton: a contradiction. Therefore, $\overline{\lambda} \neq 0, \overline{\lambda}_1 \neq 0$.

According to (45) and (46) and $\overline{\lambda} \neq 0, \overline{\lambda}_1 \neq 0$, we obtain from (84) and (85) that

$$\int_{\mu_{i+1}}^{t} \pi n_1 \int_0^1 \sin(x\pi n_1) z(x,\tau) dx d\tau = -\frac{1}{\overline{\lambda}_1} \int_{\mu_{i+1}}^{t} \pi n_1 \int_0^1 \sin(x\pi n_1) w(x,\tau) dx d\tau, \quad n_1 \in I_1, \tag{B3}$$

$$\int_{\mu_{i+1}}^{t} \pi n_2 \int_0^1 \sin(x\pi n_2) z(x,\tau) dx d\tau = -\overline{\lambda} \int_{\mu_{i+1}}^{t} \pi n_2 \int_0^1 \sin(x\pi n_2) w(x,\tau) dx d\tau, \quad n_2 \in I_2, \tag{B4}$$

for $t \in [\mu_{i+1}, \tau_{i+1}]$. According to the continuity of the mappings $\tau \to \int_0^1 \sin(x\pi n) z[\tau] dx$ and $\tau \to \int_0^1 \sin(x\pi n) w[\tau] dx$, $n \in \mathbb{N}$ (a consequence of the fact that $z \in C^0([\mu_{i+1}, \tau_{i+1}]; L^2(0,1))$ and $w \in C^0([\mu_{i+1}, \tau_{i+1}]; L^2(0,1))$, (B3) and (B4) imply

$$\int_0^1 \sin(x\pi n_1)(z(x,\tau) + \frac{1}{\overline{\lambda}_1} w(x,\tau)) dx = 0, \quad n_1 \in I_1, \tag{B5}$$

$$\int_0^1 \sin(x\pi n_2)(z(x,\tau) + \overline{\lambda} w(x,\tau)) dx = 0, \quad n_2 \in I_2, \tag{B6}$$

for $\tau \in [\mu_{i+1}, \tau_{i+1}]$. We then prove $I_1 = I_2$ in (B5), (B6). If I_2 includes elements not belonging to I_1, there exists $n_2 \in I_2$ with $n_2 \notin I_1$ such that $\int_0^1 \sin(x\pi n_2) z(x,\tau) dx = 0$ on $\tau \in [\mu_{i+1}, \tau_{i+1}]$ due to the fact that $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) = 0$ for $n \notin I_1$ with recalling (55) and (45), and then

$$\int_0^1 \sin(x\pi n_2)(z(x,\tau) + \overline{\lambda} w(x,\tau)) dx = \int_0^1 \sin(x\pi n_2) \overline{\lambda} w(x,\tau) dx, \tag{B7}$$

which is not identically zero on $\tau \in [\mu_{i+1}, \tau_{i+1}]$ because of $Q_{n_2,3}(\mu_{i+1}, \tau_{i+1}) \neq 0$ together with (57) and (46) and $\overline{\lambda} \neq 0$. This contradicts (B6). Similarly, if I_1 includes elements not belonging to I_2, there exists $n_1 \in I_1$ with $n_1 \notin I_2$ such that

$$\int_0^1 \sin(x\pi n_1)(z(x,\tau) + \frac{1}{\overline{\lambda}_1} w(x,\tau)) dx = \int_0^1 \sin(x\pi n_1) z(x,\tau) dx, \tag{B8}$$

which is not identically zero on $\tau \in [\mu_{i+1}, \tau_{i+1}]$ because of $Q_{n_1,1}(\mu_{i+1}, \tau_{i+1}) \neq 0$ together with (55) and (45), where $\int_0^1 \sin(x\pi n_1) \frac{1}{\overline{\lambda}_1} w(x,\tau) dx = 0$ on $\tau \in [\mu_{i+1}, \tau_{i+1}]$ is due to the fact that $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) = 0$ for $n \notin I_2$ with recalling (57) and (46). This contradicts (B5). Therefore, we conclude $I_1 = I_2$ in (B5) and (B6).

We then prove $\overline{\lambda} = \frac{1}{\overline{\lambda}_1}$ by contradiction. If $\overline{\lambda} - \frac{1}{\overline{\lambda}_1} \neq 0$, recalling $I_1 = I_2$ and (B6), then we obtain

$$\begin{aligned}\int_0^1 \sin(x\pi n_1)(z(x,\tau) + \frac{1}{\overline{\lambda}_1} w(x,\tau))dx &= \int_0^1 \sin(x\pi n_1)(z(x,\tau) + (\frac{1}{\overline{\lambda}_1} - \overline{\lambda} + \overline{\lambda})w(x,\tau))dx \\ &= \int_0^1 \sin(x\pi n_2)(z(x,\tau) + \overline{\lambda} w(x,\tau))dx + \int_0^1 \sin(x\pi n_2)(\frac{1}{\overline{\lambda}_1} - \overline{\lambda})w(x,\tau)dx \\ &= \left(\frac{1}{\overline{\lambda}_1} - \overline{\lambda}\right) \int_0^1 \sin(x\pi n_2) w(x,\tau)dx, \end{aligned} \tag{B9}$$

which is not identically zero for $\tau \in [\mu_{i+1}, \tau_{i+1}]$ because of $Q_{n_2,3}(\mu_{i+1}, \tau_{i+1}) \neq 0$ with (57), (46), which contradicts (B5). Therefore $\overline{\lambda} - \frac{1}{\overline{\lambda}_1} = 0$. Claim 1 is proven.

APPENDIX C. PROOF OF CLAIM 2

According to (45), (46), the equations (84), (85) ($\overline{\lambda} \neq 0, \overline{\lambda}_1 \neq 0$ and $\overline{\lambda} = \frac{1}{\overline{\lambda}_1}$) are equivalent to

$$\int_0^1 \sin(x\pi n)(z(x,\tau) + \overline{\lambda} w(x,\tau))dx = 0, \ n \in I_2 \cup I_1. \tag{C1}$$

If $\mathbb{N} = I_2 \cup I_1$, it means that (C1) holds for all $n \in \mathbb{N}$. If $I_2 \cup I_1 \subset \mathbb{N}$, recalling the definitions of I_1, I_2, we know that $\int_0^1 \sin(x\pi n) z(x,\tau)dx = \int_0^1 \sin(x\pi n) w(x,\tau)dx = 0$ for $n \in \complement_{\mathbb{N}}\{I_1 \cup I_2\}$ on $\tau \in [\mu_{i+1}, \tau_{i+1}]$, and thus (C1) is equivalent to

$$\int_0^1 \sin(x\pi n)(z(x,\tau) + \overline{\lambda} w(x,\tau))dx = 0, \quad n = 1, 2, \dots \tag{C2}$$

on $\tau \in [\mu_{i+1}, \tau_{i+1}]$. Since the set $\{\sqrt{2}\sin(n\pi x) : n = 1, 2, \dots\}$ is an orthonormal basis of $L^2(0,1)$, if (C2) holds, it follows that $z(x,t) + \overline{\lambda} w(x,t) = 0$ for $t \in [\mu_{i+1}, \tau_{i+1}]$.

If $z(x,t) + \overline{\lambda} w(x,t) = 0$ for $t \in [\mu_{i+1}, \tau_{i+1}]$, then (C2), and (84), (85) ($\overline{\lambda} \neq 0, \overline{\lambda}_1 \neq 0$ and $\overline{\lambda} = \frac{1}{\overline{\lambda}_1}$), naturally hold. Claim 2 is proven.

APPENDIX D. PROOF OF CLAIM 3

The necessary condition for the equation $z(x,t) + \overline{\lambda} w(x,t) = 0 \left(\overline{\lambda} \neq 0\right)$ to hold on $x \in [0,1], t \in [\mu_{i+1}, \tau_{i+1}]$ is that $z(x,t), w(x,t)$ are kept constant on $x \in [0,1], t \in [\mu_{i+1}, \tau_{i+1}]$ excluding finitely many possible points of discontinuity, that is, $w(x,t) = M$ and $z(x,t) = -\overline{\lambda} M$ on $x \in [0,1], t \in [\mu_{i+1}, \tau_{i+1}]$ excluding finitely many possible points of discontinuity, where M is a nonzero constant (because $z[t], w[t]$ are not identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$). We prove this by contradiction next.

Taking a spatial interval $[\underline{x}_1, \overline{x}_2] \in [0,1]$ with $\overline{x}_2 - \underline{x}_1 \leq (q_1 + q_2)(\tau_{i+1} - \mu_{i+1})$ (the position of the interval $[\underline{x}_1, \overline{x}_2]$ is arbitrary on $[0,1]$, and $(\underline{x}_1, \mu_{i+1})$, $(\overline{x}_2, \mu_{i+1})$ are not points of discontinuity of the functions $w(x,t), z(x,t)$), suppose that there exist x_a, x_b (without loss of generality we assume $x_a < x_b$) in the interval $[\underline{x}_1, \overline{x}_2]$ with $w(x_a, \mu_{i+1}) \neq w(x_b, \mu_{i+1})$, where (x_a, μ_{i+1}), (x_b, μ_{i+1}) are not points of discontinuity of the functions $w(x,t), z(x,t)$. Also we know that $z(x_a, \mu_{i+1}) = -\overline{\lambda} w(x_a, \mu_{i+1})$ according to $z(x,t) + \overline{\lambda} w(x,t) = 0$ always holding on $x \in [0,1], t \in [\mu_{i+1}, \tau_{i+1}]$. Because the state of the w-PDE propagates from $x = 1$ to $x = 0$, and the state of the z-PDE propagates from $x = 0$ to $x = 1$, with the respective propagation speeds q_1, q_2, according to the statement in p. 60 in Reference 52, which indicates that the system (2), (3) is equivalent to a pair of scalar delay equations even if the solutions are not differentiable and even not continuous with respect to t and x, we get the following relationships:

$$w(x_b - s_1 q_2, \mu_{i+1} + s_1) = w(x_b, \mu_{i+1}), \tag{D1}$$

$$z(x_a + s_1 q_1, \mu_{i+1} + s_1) = z(x_a, \mu_{i+1}), \tag{D2}$$

for $s_1 \in [0, \min\{\frac{x_b}{q_2}, \frac{1-x_a}{q_1}\}]$, where $(x_b - s_1 q_2, \mu_{i+1} + s_1)$, $(x_a + s_1 q_1, \mu_{i+1} + s_1)$ are not the points of discontinuity because (x_b, μ_{i+1}), (x_a, μ_{i+1}) are not the points of discontinuity. There exists a $s_1 = \frac{x_b - x_a}{q_1 + q_2}$ such that $x_b - s_1 q_2 = x_a + s_1 q_1 = x_c$, and then we obtain

$$w(x_c, t_c) = w(x_b, \mu_{i+1}), \quad z(x_c, t_c) = z(x_a, \mu_{i+1}), \tag{D3}$$

where $x_c = \frac{q_1 x_b + q_2 x_a}{q_1 + q_2} \in (x_a, x_b)$, $t_c = \mu_{i+1} + \frac{x_b - x_a}{q_1 + q_2} \in (\mu_{i+1}, \tau_{i+1}]$ recalling $x_b - x_a \le \overline{x}_2 - \underline{x}_1 \le (q_1 + q_2)(\tau_{i+1} - \mu_{i+1})$. Because

$$z(x_a, \mu_{i+1}) = -\overline{\lambda} w(x_a, \mu_{i+1}) \neq -\overline{\lambda} w(x_b, \mu_{i+1}),$$

recalling $\overline{\lambda} \neq 0$ and the hypothesis that $w(x_a, \mu_{i+1}) \neq w(x_b, \mu_{i+1})$, using (D3), we have

$$z(x_c, t_c) \neq -\overline{\lambda} w(x_c, t_c),$$

with $x_c \in [0, 1]$, $t_c \in (\mu_{i+1}, \tau_{i+1}]$: a contradiction. Therefore, the hypothesis that there exist x_a, x_b in the interval $[\underline{x}_1, \overline{x}_2]$ such that $w(x_a, \mu_{i+1}) \neq w(x_b, \mu_{i+1})$ ((x_a, μ_{i+1}), (x_b, μ_{i+1}) are not points of discontinuity) does not hold, and we then conclude $w(x, \mu_{i+1})$, $z(x, \mu_{i+1})$ are kept constant on $x \in [\underline{x}_1, \overline{x}_2]$ excluding finitely many possible points of discontinuity. Because the position of the interval $[\underline{x}_1, \overline{x}_2]$ is arbitrary on $[0, 1]$ (with $(\underline{x}_1, \mu_{i+1})$, $(\overline{x}_2, \mu_{i+1})$ are not points of discontinuity of the functions $w(x, t)$, $z(x, t)$), and then we have that $w(x, \mu_{i+1})$, $z(x, \mu_{i+1})$ are kept constant for $x \in [0, 1]$ excluding finitely many possible points of discontinuity. Taking a time increment s with $0 < s \le \frac{1}{2\max\{q_1, q_2\}}$, we have

$$w(x, \mu_{i+1} + s) = w(x + q_2 s, \mu_{i+1}) = w(x, \mu_{i+1}),$$

for $x \in [0, \frac{1}{2}]$, with excluding the points of discontinuity of the functions $w(x, t)$, $z(x, t)$ along $x \in [0, 1]$, $t = \mu_{i+1}$, where $s \in (0, \frac{1}{2\max\{q_1, q_2\}}]$ ensures $x + q_2 s \in (0, 1]$. Also we get that

$$z(x, \mu_{i+1} + s) = z(x - q_1 s, \mu_{i+1}) = z(x, \mu_{i+1}),$$

for $x \in [\frac{1}{2}, 1]$, with excluding the points of discontinuity of the functions $w(x, t)$, $z(x, t)$ along $x \in [0, 1]$, $t = \mu_{i+1}$, where $s \in (0, \frac{1}{2\max\{q_1, q_2\}}]$ ensures $x - q_1 s \in [0, 1)$. Recalling that $z(x, t) + \overline{\lambda} w(x, t) = 0$ always holds on $x \in [0, 1]$, $t \in [\mu_{i+1}, \tau_{i+1}]$, we have that z, w are kept constant in $x \in [0, 1]$, $t \in [\mu_{i+1}, \mu_{i+1} + \frac{1}{2\max\{q_1, q_2\}}]$ excluding finitely many possible points of discontinuity. If $\mu_{i+1} + \frac{1}{2\max\{q_1, q_2\}} \ge \tau_{i+1}$, we directly obtain the necessary condition for $z(x, t) + \overline{\lambda} w(x, t) = 0$ to hold on $x \in [0, 1]$, $t \in [\mu_{i+1}, \tau_{i+1}]$ mentioned at the beginning of the proof of Claim 3. If $\mu_{i+1} + \frac{1}{2\max\{q_1, q_2\}} < \tau_{i+1}$, repeatedly taking the time increments s and conducting the above process for k times, based on the fact that w, z are kept constant for $x \in [0, 1]$ at the beginning of each time increment, with excluding finitely many possible points of discontinuity, until $\mu_{i+1} + \frac{k}{2\max\{q_1, q_2\}} \ge \tau_{i+1}$, we also obtain the necessary condition for $z(x, t) + \overline{\lambda} w(x, t) = 0$ to hold on $x \in [0, 1]$, $t \in [\mu_{i+1}, \tau_{i+1}]$ mentioned at the beginning of the proof, namely, that $z(x, t)$, $w(x, t)$ are kept constant on $x \in [0, 1]$, $t \in [\mu_{i+1}, \tau_{i+1}]$, excluding finitely many possible points of discontinuity, that is,

$$w(x, t) = M, z(x, t) = -\overline{\lambda} M, \; (x, t) \in ([0, 1] \times [\mu_{i+1}, \tau_{i+1}]) \setminus I_d, \tag{D4}$$

where I_d denotes a set of finitely many possible points of discontinuity of the functions $w(x, t)$, $z(x, t)$ in $x \in [0, 1]$, $t \in [\mu_{i+1}, \tau_{i+1}]$, and where M is a nonzero constant (because $z[t]$, $w[t]$ are not identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$).

The situation that $w(x, t) = M$ and $z(x, t) = -\overline{\lambda} M$ for $(x, t) \in ([0, 1] \times [\mu_{i+1}, \tau_{i+1}]) \setminus I_d$, means $\zeta(t) = \frac{(-\overline{\lambda} + p)}{c} M$ on $t \in [\mu_{i+1}, \tau_{i+1}]$ according to (4), excluding finitely many possible points of discontinuity on $t \in [\mu_{i+1}, \tau_{i+1}]$. Recalling (1), it then must be that $(a - q_1 bc)\frac{(-\overline{\lambda} + p)}{c} + b(q_2 + q_1 p) = 0$. It follows that

$$\overline{\lambda} = \frac{cb(q_2 + q_1 p)}{(a - q_1 bc)} + p > 0, \tag{D5}$$

because the constants c, b, q_1, q_2, p and $a - q_1bc$ are positive. Inserting the control input (36) into the right boundary condition (5), recalling (15), (16) and $\zeta(t) = \frac{(-\overline{\lambda}+p)}{c}M$, a necessary condition of $w(x,t)=M$ and $z(x,t) = -\overline{\lambda}M$ for $(x,t) \in ([0,1] \times [\mu_{i+1}, \tau_{i+1}]) \setminus I_d$, is that the following equation holds

$$[q_2 + (q_1 - \hat{q}_1(\tau_i))\overline{\lambda}]M = \kappa[\hat{q}_2(\tau_i)\int_0^1 \frac{b}{\hat{q}_2(\tau_i)} e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)(1-y)}dy + \frac{\hat{q}_2(\tau_i)(-\overline{\lambda}+p)}{c(\hat{q}_1(\tau_i)p + \hat{q}_2(\tau_i))} e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)}]M, \tag{D6}$$

that is,

$$q_2 + (q_1 - \hat{q}_1(\tau_i))\left(\frac{cb(q_2+q_1p)}{a-q_1bc} + p\right) = -\kappa\left[\hat{q}_2(\tau_i)b\left(\frac{1-e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)}}{a-\hat{q}_1(\tau_i)bc} + \frac{e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)}}{\hat{q}_1(\tau_i)p+\hat{q}_2(\tau_i)}\frac{(q_2+q_1p)}{(a-q_1bc)}\right)\right], \tag{D7}$$

because of $M \neq 0$. Recalling $\hat{q}_1(0) = \underline{q}_1$ and $\hat{q}_2(0) = \underline{q}_2$, we have $0 < \underline{q}_1 \leq \hat{q}_1(\tau_i) \leq q_1$, $0 < \underline{q}_2 \leq \hat{q}_2(\tau_i) \leq q_2$ (the consequence of (61) and the fact that $\theta \in S_i$ defined by (76)), which implies $\frac{(q_2+q_1p)}{(a-q_1bc)} \geq \frac{(\hat{q}_2(\tau_i)+\hat{q}_1(\tau_i)p)}{(a-\hat{q}_1(\tau_i)bc)}$. We thus have

$$\begin{aligned}&\frac{1-e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)}}{a-\hat{q}_1(\tau_i)bc} + \frac{e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)}}{\hat{q}_1(\tau_i)p+\hat{q}_2(\tau_i)}\frac{(q_2+q_1p)}{(a-q_1bc)}\\ &\geq \frac{1-e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)}}{a-\hat{q}_1(\tau_i)bc} + \frac{e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)}}{\hat{q}_1(\tau_i)p+\hat{q}_2(\tau_i)}\frac{(\hat{q}_2(\tau_i)+\hat{q}_1(\tau_i)p)}{(a-\hat{q}_1(\tau_i)bc)}\\ &\geq \frac{1}{a-\hat{q}_1(\tau_i)bc} > 0.\end{aligned} \tag{D8}$$

Therefore, recalling the constants $c, b, q_1, q_2, p, a - q_1bc$ and $\hat{q}_1(\tau_i), \hat{q}_2(\tau_i)$ are positive, the right-hand side of (D7) is greater than zero because of $\kappa < 0$ and (D8), and the left-hand side of (D7) is also greater than zero because of $\hat{q}_1(\tau_i) \leq q_1$. The necessary condition of $w(x,t)=M$ and $z(x,t) = -\overline{\lambda}M$ for $(x,t) \in ([0,1] \times [\mu_{i+1}, \tau_{i+1}]) \setminus I_d$ with $M \neq 0$ becomes that the design parameter κ is equal to $\overline{\kappa}$ defined as

$$\overline{\kappa} = -\left(q_2 + (q_1 - \hat{q}_1(\tau_i))\left(\frac{cb(q_2+q_1p)}{a-q_1bc} + p\right)\right) \div \left[\hat{q}_2(\tau_i)b\left(\frac{1-e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)}}{a-\hat{q}_1(\tau_i)bc} + \frac{e^{\frac{1}{\hat{q}_2(\tau_i)}(a-\hat{q}_1(\tau_i)bc)}}{\hat{q}_1(\tau_i)p+\hat{q}_2(\tau_i)}\frac{(q_2+q_1p)}{(a-q_1bc)}\right)\right] < 0. \tag{D9}$$

According to (D8) and $\hat{q}_1(\tau_i) \geq \underline{q}_1, \hat{q}_2(\tau_i) \geq \underline{q}_2$, we know that $\overline{\kappa}$ defined by (D9) is in the following range

$$\frac{(a-\underline{q}_1bc)[q_2 + (q_1 - \underline{q}_1)(\frac{cb(q_2+q_1p)}{a-q_1bc} + p)]}{-\underline{q}_2 b} \leq \overline{\kappa} \leq 0. \tag{D10}$$

Recalling the first term in (12), we know that $\kappa \neq \overline{\kappa}$. We thus conclude that $w(x,t)=M$ and $z(x,t) = -\overline{\lambda}M$ with $M \neq 0$ on $(x,t) \in ([0,1] \times [\mu_{i+1}, \tau_{i+1}]) \setminus I_d$ does not hold. Claim 3 is proven.

APPENDIX E. PROOF OF CLAIM 4

We first prove sufficiency. If $z[t]$ (or $w[t]$) are not identically zero for $t = [0, \lim_{i\to\infty}(\tau_i))$, there exists an interval $[\mu_{i+1}, \tau_{i+1}]$ on which $z[t]$ (or $w[t]$) are not identically zero. It follows that $\hat{q}_1(\tau_{i+1}) = q_1$ (or $\hat{q}_2(\tau_{i+1}) = q_2$) recalling Lemma 3.

Next, we prove necessity. When $\hat{q}_1(0) \neq q_1$ (or $\hat{q}_2(0) \neq q_2$), if the estimate reaches the true value at an instant τ_{i+1}, that is, $\hat{q}_1(\tau_{i+1}) = q_1$ (or $\hat{q}_2(\tau_{i+1}) = q_2$), it follows there exists $n \in \mathbb{N}$ such that $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) \neq 0$ (or $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) \neq 0$). (This is true because, if $Q_{n,1}(\mu_{i+1}, \tau_{i+1}) = 0$ (or $Q_{n,3}(\mu_{i+1}, \tau_{i+1}) = 0$) for all $n \in \mathbb{N}$, it would also be true that $g_{n,1}(t, \mu_{i+1}) = 0$ (or $g_{n,2}(t, \mu_{i+1}) = 0$) for all $n \in \mathbb{N}$ on $t \in [\mu_{i+1}, \tau_{i+1}]$, according to (55) and (57). It follows that $Q_{n,2}(\mu_{i+1}, \tau_{i+1}) = 0$, $H_{n,1}(\mu_{i+1}, \tau_{i+1}) = 0$ (or $Q_{n,2}(\mu_{i+1}, \tau_{i+1}) = 0, H_{n,2}(\mu_{i+1}, \tau_{i+1}) = 0$) for all $n \in \mathbb{N}$ according to (53)–(57). Consequently, we have from (61) that $\hat{q}_1(\tau_{i+1}) = \hat{q}_1(\tau_i) \neq q_1$ (or $\hat{q}_2(\tau_{i+1}) = \hat{q}_2(\tau_i) \neq q_2$) by recalling (59)–(60)). We then conclude that $z[t]$ (or $w[t]$) are not identically zero on $t \in [\mu_{i+1}, \tau_{i+1}]$ according to Lemma 2. That is, $z[t]$ (or $w[t]$) are not identically zero on $t = [0, \lim_{i\to\infty}(\tau_i))$.

The proof of Claim 4 is complete.

APPENDIX F. PROOF OF CLAIM 4

We prove this claim by estimating the largest convergence time of parameter estimates τ_ε in various situations of initial conditions $z[0], w[0], \zeta(0)$. Inserting (36) into (5), we get

$$q_2 w(1,t) = \hat{q}_2(t) \int_0^1 \phi(1, y; \hat{\theta}(t)) w(y,t) dy + \hat{q}_2(t)\lambda(1; \hat{\theta}(t))\zeta(t) + \tilde{q}_1(t) z(1,t). \tag{F1}$$

Case 1: $z[0] \neq 0, w[0] = 0, \zeta(0) = 0$. According to Lemma 3, we have $\hat{q}_1(\tau_1) = q_1$ and $\tilde{q}_1(t) \equiv 0$ for $t \geq \tau_1$. If $w[t] \equiv 0$ on $t \in [0, \tau_1]$, then $w[t]$ and $\zeta(t)$ are identically zero on $t \geq 0$ according to (1), (3) and (F1) with $\tilde{q}_1(t) \equiv 0$ for $t \geq \tau_1$. If $\hat{q}_2(0) \neq q_2$, it follows that $\hat{q}_2(t)$ cannot reach the true value q_2 according to Claim 4 with property 1: a contradiction with the fact that $\hat{q}_2(t)$ would reach q_2 in finite time. Thus $w[t]$ is not identically zero on $t \in [0, \tau_1]$ if $\hat{q}_2(0) \neq q_2$. It follows that $\hat{q}_2(t)$ can reach q_2 not later than τ_1 according to Lemma 3. It is obtained from (38) that the dwell time is less than or equal to T. Therefore $\tau_\varepsilon \leq T$.

Case 2: $w[0] \neq 0, z[0] = 0, \zeta(0) = 0$. The maximum time taken by the nonzero values of $w[0]$ propagate to $x = 0$ and enter $z(0,t)$ is $\frac{1}{q_2}$. Therefore, the estimate $\hat{q}_1(t)$ would reach the true value q_1 not later than $\tau_f = \min\{\tau_f : f \in Z_+, \tau_f > \frac{1}{q_2}\}$ according to Lemma 3. Because of $w[0] \neq 0$, we have $\hat{q}_2(\tau_1) = q_2$. It follows that $\tau_\varepsilon \leq \frac{1}{q_2} + T$ because the dwell time is less than or equal to T.

Case 3: $\zeta(0) \neq 0, z[0] = 0, w[0] = 0$. According to (F1) and (4), we know that $w[t], z[t]$ are not identically zero on $t \in [0, \tau_1]$, which implies that the estimates $\hat{\theta}(t)$ reach the true values θ not later than τ_1 according to Lemma 3. Therefore $\tau_\varepsilon \leq \tau_1 \leq T$.

Case 4: $\zeta(0) \neq 0, w[0] \neq 0, z[0] = 0$. The necessary condition of the fact that $z[t]$ is identically zero (i.e., $w(0,t) = \frac{c}{p}\zeta(t)$ always holds) for $t \in [0, \tau_f]$ where $\tau_f = \min\{\tau_f : f \in Z_+, \tau_f > \frac{1}{q_2}\}$ is $\kappa > 0$, according to the analysis in Case 2 in the proof of the portion 1 of the theorem. Recalling $\kappa < 0$ in (12), we know that $z[t]$ is not identically zero on $t \in [0, \tau_f]$, which implies that the estimate $\hat{q}_1(t)$ reaches the true value q_1 not later than τ_f according to Lemma 3. Because of $w[0] \neq 0$, we have that $\hat{q}_2(\tau_1) = q_2$. Therefore $\tau_\varepsilon \leq \frac{1}{q_2} + T$.

Case 5: $\zeta(0) \neq 0, z[0] \neq 0, w[0] = 0$. According to Lemma 3, we have that $\hat{q}_1(\tau_1) = q_1$ and $\tilde{q}_1(t) \equiv 0$ for $t \geq \tau_1$. If $w[t] \equiv 0$ on $t \in [0, \tau_2]$, it follows from (1) that $\zeta(t) = e^{(a - q_1 bc)t}\zeta(0)$ is not identically zero on $t \in [\tau_1, \tau_2]$. It is obtained from (F1) that $w(1,t)$ is not identically zero on $t \in [\tau_1, \tau_2]$: a contradiction. Therefore, $w[t]$ are not identically zero on $t \in [0, \tau_2]$, which implies that the estimate $\hat{q}_2(t)$ reaches the true value q_2 not later than τ_2 according to Lemma 3. Therefore, we have that $\tau_\varepsilon \leq \tau_2 \leq 2T$.

Case 6: $\zeta(0) = 0, z[0] \neq 0, w[0] \neq 0$ and Case 7: $\zeta(0) \neq 0, z[0] \neq 0, w[0] \neq 0$. According to Lemma 3, we have that $\tau_\varepsilon \leq \tau_1 \leq T$.

Case 8: $z[0] = 0, w[0] = 0, \zeta(0) = 0$. According to the plant (1)–(5) with the control input (36), we know that $z[t], w[t], \zeta(t)$ are identically zero for $t \geq 0$. The estimates reach the true values in finite time only when $\hat{q}_1(0) = q_1, \hat{q}_2(0) = q_2$, that is, $\tau_\varepsilon = 0$.

In summary, we have proved for all eight cases that $\tau_\varepsilon \leq \max\{\frac{1}{q_2} + T, 2T\}$. This completes the proof of Claim 5.